2023 August Qualifying Exam

Day 1

1. Let $Y_1, \ldots, Y_n$ be a random sample from the distribution with pdf given by

$$f_Y(y;\beta) \begin{cases} \frac{1}{(k-1)!\beta^k}\left(\frac{1}{y}\right)^{k+1}\exp\left(-\frac{1}{y\beta}\right), & y > 0 \\ 0, & y \leq 0, \end{cases}$$

where $\beta > 0$ is unknown and k is a known positive integer.

(a) Find the maximum likelihood estimator $\hat{\beta}_n$ of β.

(b) Determine whether $\hat{\beta}_n$ is the UMVUE for β. Give careful arguments to support your answer.

(c) Give an interval in which $\hat{\beta}_n$ will fall with probability approximately 0.95 provided n is large.

(d) Give a lower bound for the variance of any unbiased estimator of $\tau(\beta) = \frac{1}{(k+1)\beta}$ (the mode of f_Y).

(e) Give an estimator of $\tau(\beta)$ of which the variance approaches the smallest possible variance as $n \to \infty$.

(f) Suppose we wish to test H_0: $\beta \leq 1$ versus H_1: $\beta > 1$. Give a sequence of values c_n such that the decision rules $\hat{\beta}_n > c_n$ define tests with limiting size equal to α as $n \to \infty$. Your c_n values should involve a quantile of the standard Normal distribution.

(g) Using the decision rule $\hat{\beta}_n > c_n$ with c_n as determined in the previous part, give a formula for the approximate sample size required to reject H_0: $\beta \leq 1$ in favor of H_1: $\beta > 1$ with probability at least γ^* when the true value of the parameter is $\beta^* > 1$.

(h) Give the formula for an asymptotic $(1-\alpha) \times 100\%$ confidence interval for β that arises from inverting the score test of H_0: $\beta = \beta_0$ versus H_1: $\beta \neq \beta_0$. Recall that the score test statistic is $[S(\beta_0; Y_1, \ldots, Y_n)]^2 / I_n(\beta_0)$.

2. Let $X_1, \ldots, X_n$ be iid realizations of a random variable $X \in (0, \infty)$ with $\mathbb{E}X = \mu$ and $\operatorname{Var} X = \mu + \theta\mu$.

   (a) Suppose the distribution of X is unknown.

      i. Give the method of moments estimators of μ and θ based on $X_1, \ldots, X_n$.

      ii. Give conditions under which the method of moments estimators are consistent and carefully argue why the conditions imply consistency.

      iii. Give an asymptotic size-α confidence interval for μ. Give a justification for the interval.

   (b) Now suppose $X|\Lambda \sim \text{Poisson}(\mu\Lambda)$, where Λ has a gamma distribution.

      i. Choose gamma parameters for the distribution of Λ such that $\mathbb{E}X = \mu$ and $\operatorname{Var} X = \mu + \theta\mu$.

      ii. Find the marginal probability mass function of X.

      iii. Give the limiting marginal distribution of X as $\theta \to 0$.

      iv. Under this hierarchical model it has been shown that as $\theta \to 0$ the quantity $\sum_{i=1}^{n}(X_i - \bar{X}_n)^2/\bar{X}_n$ becomes distributed approximately as a χ^2_{n-1} random variable, provided n is not too small. Use this fact to find a decision rule for whether you should believe $\operatorname{Var} X = \mu$ or $\operatorname{Var} X > \mu$ based on the data $X_1, \ldots, X_n$. State how often your rule will allow you to falsely conclude $\operatorname{Var} X > \mu$.

3. Let $X_1, \ldots, X_n$ be iid Normal(θ, θ^2) random variables where θ is unknown and $\theta \neq 0$. The likelihood function for θ is given by

$$\mathcal{L}(\theta) = (\sqrt{2\pi}|\theta|)^{-n} \exp\left\{-\frac{\sum_{i=1}^n (X_i - \theta)^2}{2\theta^2}\right\}.$$

(a) Show that the maximum likelihood estimator $\hat{\theta}$ of θ takes the form

$$\hat{\theta} = \hat{\theta}_+ I\{\log \mathcal{L}(\hat{\theta}_+) \geq \log \mathcal{L}(\hat{\theta}_-)\} + \hat{\theta}_- I\{\log \mathcal{L}(\hat{\theta}_-) \geq \log \mathcal{L}(\hat{\theta}_+)\},$$

where

$$\hat{\theta}_+ = \frac{1}{2}(-\bar{X} + \sqrt{5\bar{X}^2 + 4T^2}) \quad \text{and} \quad \hat{\theta}_- = \frac{1}{2}(-\bar{X} - \sqrt{5\bar{X}^2 + 4T^2}),$$

in which $\bar{X}$ is the sample mean and $T^2 = n^{-1}\sum_{i=1}^n (X_i - \bar{X})^2$.

(b) When $\theta > 0$, show that $\hat{\theta}_+$ is a consistent estimator of θ but $\hat{\theta}_-$ is inconsistent. Obtain a similar result for the $\theta < 0$ case.

(c) Obtain the asymptotic distribution of $\hat{\theta}$ and find the asymptotic relative efficiency of $\hat{\theta}$ with respect to the sample mean $\bar{X}$.

(d) Suppose $\theta > 0$ and $\mathbb{E}X_i = \theta$ and $\operatorname{Var} X_i = \theta^2$, but the distribution of X_i is not necessarily normal. Furthermore, suppose that $\operatorname{Var}(X_i^2) = a\theta^4$ and $\operatorname{Cov}(X_i, X_i^2) = b\theta^3$ for some constants a and b, where $a > 0$. Obtain the asymptotic distribution of $\hat{\theta}_+$.