2022 August Qualifying Exam

Day 1

1. Let $W_1, \ldots, W_n \overset{\text{ind}}{\sim} f_W(w; \lambda) = \frac{3w^2}{\lambda} e^{-w^3/\lambda} \mathbf{1}(w > 0)$ for some $\lambda > 0$.
   (a) i. Find the method of moments estimator $\hat{\lambda}_{\text{MoM}}$ of λ based on $W_1, \ldots, W_n$.
      ii. Give the asymptotic behavior of $\sqrt{n}(\hat{\lambda}_{\text{MoM}} - \lambda)$ as $n \to \infty$.
   (b) i. Find the maximum likelihood estimator $\hat{\lambda}_{\text{MLE}}$ for λ based on $W_1, \ldots, W_n$.
      ii. Give the asymptotic behavior of $\sqrt{n}(\hat{\lambda}_{\text{MLE}} - \lambda)$ as $n \to \infty$.
      iii. Give an asymptotic $(1 - \alpha) \times 100\%$ confidence interval for $\log \lambda$.
      iv. Give the likelihood ratio test of H_0: $\lambda \leq \lambda_0$ versus H_1: $\lambda > \lambda_0$. Calibrate the rejection region so that the test has size α.
   (c) i. Give a pivotal quantity for the parameter λ.
      ii. Based on your pivotal quantity, construct an exact $(1 - \alpha) \times 100\%$ confidence interval for λ.
   (d) Give the UMVUE for λ. Give a detailed justification.

2. Suppose there are n ultrasound images taken on a random sample of babies of the same gestational age, from which it is of interest to estimate the mean size of some feature in the population. In measuring the feature, ultrasound techs are liable to some error, meaning that different ultrasound techs measuring the same feature from the same image may record different values, and a single tech may record different values when presented with the same image twice. Suppose a number of ultrasound techs are recruited to measure a feature in n images such that two techs are assigned to each image. Assume the feature sizes $X_1, \ldots, X_n$ are Normally distributed with mean μ and variance σ_X^2. Furthermore, denote the recorded measurements from image i as (Y_{i1}, Y_{i2}) and assume $Y_{i1} = X_i + \varepsilon_{i1}$ and $Y_{i2} = X_i + \varepsilon_{i2}$ for each $i = 1, \ldots, n$, where $\varepsilon_{11}, \varepsilon_{12}, \ldots, \varepsilon_{n1}, \varepsilon_{n2}$ are independent Normal random variables with mean 0 and variance σ_ε^2. The parameters μ, σ_X^2, and σ_ε^2 are unknown.
   (a) Show that the estimators

$$\hat{\mu} = \frac{1}{2n} \sum_{i=1}^{n} \sum_{j=1}^{2} Y_{ij}, \quad \hat{\sigma}_\varepsilon^2 = \frac{1}{2n} \sum_{i=1}^{n} (Y_{i1} - Y_{i2})^2, \quad \text{and} \quad \hat{\sigma}_X^2 = \frac{1}{n-1} \sum_{i=1}^{n} (\bar{Y}_{i.} - \hat{\mu})^2 - \frac{1}{2} \hat{\sigma}_\varepsilon^2$$

   are unbiased estimators of μ, σ_ε^2, and σ_X^2, respectively, where $\bar{Y}_{i.} = (Y_{i1} + Y_{i2})/2$ for $i = 1, \ldots, n$.
   (b) Give $\operatorname{Var} \hat{\mu}$.
   (c) Obtain a pivotal quantity for σ_ε^2 and use it to construct a $(1 - \alpha) \times 100\%$ confidence interval for σ_ε^2.

(d) Obtain a pivotal quantity for μ (note that σ_X^2 and σ_ε^2 are unknown) and use it to construct an exact $(1-\alpha)\times 100\%$ confidence interval for μ. Note that the quantities $\sum_{i=1}^n \sum_{j=1}^2 Y_{ij}$ and $\sum_{i=1}^n (\bar{Y}_{i.} - \hat{\mu})^2$ are independent.

(e) Suggest a modification to your confidence intervals in parts (c) and (d) so the event that they both contain their targets occurs with probability at least $1-\alpha$.

(f) Write down the likelihood function for μ, σ_X^2, and σ_ε^2 based on the observed data (Y_{i1}, Y_{i2}), $i = 1, \ldots, n$. Note that each pair (Y_{i1}, Y_{i2}), $i = 1, \ldots, n$, follows a bivariate Normal distribution.

(g) Find the maximum likelihood estimator of μ.

3. Let Y follow a mixture distribution with the density function

$$f(y) = p_1 f_1(y; \mu_1, \sigma_1) + p_2 f_2(y; \mu_2, \sigma_2) + \cdots + p_m f_m(y; \mu_m, \sigma_m), \tag{1}$$

where $p_1, \ldots, p_m$ are positive values with $\sum_{j=1}^m p_j = 1$ and $f_j(y; \mu_j, \sigma_j)$ is a density function with mean μ_j and standard deviation σ_j, for $j = 1, \ldots, m$.

(a) Verify that (1) is a valid density function.

(b) Find $\mathbb{E}Y$ and $\operatorname{Var} Y$.

(c) To better understand the mixture model, we can use the following hierarchical structure.

$$\begin{aligned} Y|\mathbf{X} = (X_1, X_2, \ldots, X_m) &\sim \prod_{j=1}^m (f_j(y; \mu_j, \sigma_j))^{X_j} \\ \mathbf{X} = (X_1, X_2, \ldots, X_m) &\sim \text{Multinomial}(1, p_1, p_2, \ldots, p_m). \end{aligned} \tag{2}$$

Note that $\mathbf{X}$ is an unobserved latent random vector. Show that the marginal distribution of Y based on the hierarchical structure (2) is the mixture distribution (1).

(d) Based on the hierarchical structure in (2), derive the conditional distribution of $\mathbf{X}$ given Y. Specify the parameters of the conditional distribution clearly if $f_j(y; \mu_j, \sigma_j)$'s are normal densities.

(e) Now let $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_n)$ be a random sample such that

$$\begin{aligned} Y_i|\mathbf{X}_i = (X_{i1}, X_{i2}, \ldots, X_{im}) &\sim \prod_{j=1}^m (f_j(y_i; \mu_j, \sigma_j))^{X_{ij}} \\ \mathbf{X}_i = (X_{i1}, X_{i2}, \ldots, X_{im}) &\sim \text{Multinomial}(1, p_1, p_2, \ldots, p_m), \end{aligned} \tag{3}$$

for $i = 1, \ldots, n$. Note that the $\mathbf{X}_i$'s are unobserved latent random vectors. Assume the μ_j's and $\sigma_j's$ are known constants, but $p_1, p_2, \ldots, p_m$ are random parameters. Let the prior distribution of $(p_1, p_2, \ldots, p_m)$ be the Dirichlet distribution. That is, let

$$(p_1, p_2, \ldots, p_m) \sim \pi(p_1, p_2, \ldots, p_m) = \frac{\prod_{j=1}^m \Gamma(\alpha_j)}{\Gamma(\sum_{j=1}^m \alpha_j)} \prod_{j=1}^m p_j^{\alpha_j - 1}, \quad \alpha_j > 0. \tag{4}$$

Given a random sample $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ observed, our major interest is to access the posterior distribution of $(p_1, p_2, \ldots, p_m)$, which we denote by $\pi((p_1, p_2, \ldots, p_m)|\mathbf{y})$. However, it is difficult to derive it directly. Instead consider deriving the full set of conditional posterior distributions:

i. Based on the hierarchical model (3) and prior (4), derive and give the name of the conditional posterior distribution of $(p_1, p_2, \ldots, p_m)$ conditional on $\mathbf{y}$ and $(\mathbf{X}_1, \ldots, \mathbf{X}_n) = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$, denoted by $\pi((p_1, p_2, \ldots, p_m)|\mathbf{y}, (\mathbf{x}_1, \ldots, \mathbf{x}_n))$.

ii. Based on the hierarchical model (3) and prior (4), derive and give the name the conditional posterior distribution of $\mathbf{X}_i = (X_{i1}, X_{i2}, \ldots, X_{im})$, conditional on $\mathbf{y}$ and $(p_1, \ldots, p_m)$, which we denote by $\pi((x_{i1}, x_{i2}, , x_{im})|\mathbf{y}, (p_1, p_2, \ldots, p_m))$, for $i = 1, \ldots, n$.