Appendix A: Expectation, Variance, Covariance and Correlation

\[ E[h(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x, y)f(x, y)dxdy \]  
\hspace{1cm} (2.A.1)

As a corollary to Equation (2.A.1), we easily obtain the important result

\[ E(aX + bY + c) = aE(X) + bE(Y) + c \]  
\hspace{1cm} (2.A.2)

We also have

\[ E(XY) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xyf(x, y)dxdy \]  
\hspace{1cm} (2.A.3)

The variance of a random variable \( X \) is defined as

\[ Var(X) = E\{[X - E(X)]^2\} \]  
\hspace{1cm} (2.A.4)

(provided \( E(X^2) \) exists). The variance of \( X \) is often denoted by \( \sigma^2 \) or \( \sigma_X^2 \).

**Properties of Variance**

\[ Var(X) \geq 0 \]  
\hspace{1cm} (2.A.5)

\[ Var(a + bX) = b^2 Var(X) \]  
\hspace{1cm} (2.A.6)

If \( X \) and \( Y \) are independent, then

\[ Var(X + Y) = Var(X) + Var(Y) \]  
\hspace{1cm} (2.A.7)

In general, it may be shown that

\[ Var(X) = E(X^2) - [E(X)]^2 \]  
\hspace{1cm} (2.A.8)

The positive square root of the variance of \( X \) is called the **standard deviation** of \( X \) and is often denoted by \( \sigma \) or \( \sigma_X \). The random variable \( (X - \mu_X)/\sigma_X \) is called the **standardized version** of \( X \). The mean and standard deviation of a standardized variable are always zero and one, respectively.

The **covariance** of \( X \) and \( Y \) is defined as

\[ Cov(X, Y) = E[(X - \mu_X)(Y - \mu_Y)] \]

**Properties of Covariance**

\[ Cov(a + bX, c + dY) = bdCov(X, Y) \]  
\hspace{1cm} (2.A.9)

\[ Var(X + Y) = Var(X) + Var(Y) + 2Cov(X, Y) \]  
\hspace{1cm} (2.A.10)

\[ Cov(X + Y, Z) = Cov(X, Z) + Cov(Y, Z) \]  
\hspace{1cm} (2.A.11)

\[ Cov(X, X) = Var(X) \]  
\hspace{1cm} (2.A.12)

\[ Cov(X, Y) = Cov(Y, X) \]  
\hspace{1cm} (2.A.13)

If \( X \) and \( Y \) are independent,

\[ Cov(X, Y) = 0 \]  
\hspace{1cm} (2.A.14)
The correlation coefficient of \(X\) and \(Y\), denoted by \(\text{Corr}(X, Y)\) or \(p\), is defined as

\[
p = \text{Corr}(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}}
\]

Alternatively, if \(X^*\) is a standardized \(X\) and \(Y^*\) is a standardized \(Y\), then \(p = E(X^*Y^*)\).

**Properties of Correlation**

\[-1 \leq \text{Corr}(X, Y) \leq 1\]  \hspace{1cm} (2.A.15)

\[
\text{Corr}(a + bX, c + dY) = \text{sign}(bd)\text{Corr}(X, Y)
\]

where \(\text{sign}(bd) = \begin{cases} 1 & \text{if } bd > 0 \\ 0 & \text{if } bd = 0 \\ -1 & \text{if } bd < 0 \end{cases}\)  \hspace{1cm} (2.A.16)

\[
\text{Corr}(X, Y) = \pm 1 \text{ if and only if there are constants } a \text{ and } b \text{ such that } Pr(Y = a + bX) = 1.
\]

**More Useful Properties:**

\[
\text{var}(X - Y) = \text{var}(X) + \text{var}(Y) - 2\text{cov}(X, Y)
\]

\[
\text{cov}(W + X, Y + Z) = \text{cov}(W, Y) + \text{cov}(W, Z) + \text{cov}(X, Y) + \text{cov}(X, Z)
\]

\[
\text{cov}(aW + bX, cY + dZ) = (ac)\text{cov}(W, Y) + (ad)\text{cov}(W, Z) + (bc)\text{cov}(X, Y) + (bd)\text{cov}(X, Z)
\]

\[
\text{cov}(X, Y) = E(XY) - E(X)E(Y)
\]

If \(X\) and \(Y\) are independent, then

\[
E(XY) = E(X)E(Y)
\]