STAT 521 – Spring 2019 – HW 4

Note: You must show your work wherever possible! Problems 1 through 7 are required for everyone; problem 8 is required for graduate students and optional (extra credit) for undergraduates.

1. An MCMC Exercise: Suppose you wish to draw a sample from a distribution with the following pdf:
$$f(x) = 0.5\exp(-|x|),$$
where x may be any real number. [Here, $\exp(\cdot)$ is the exponential function (the number e raised to a power) and $|\cdot|$ is the absolute value function. These functions are coded as `exp()` and `abs()` in R.] Provide R code that will sample from this distribution using the Metropolis-Hastings algorithm. Run the code in R, estimate the mean and variance of the target distribution, and report your estimates. Also discuss your acceptance rate.

2. The time T required to repair a machine is an exponential random variable with mean 0.5 hours.

   (a) What is the probability that a repair time is greater than 0.5 hours?

   (b) What is the probability that a repair takes at least 12.5 hours, given that its duration is greater than 12 hours?

3. Suppose that you arrive at a single-teller bank to find five other customers in the bank, one being served and the other four waiting in line. You join the end of the line. If the service times are all exponential with **rate** μ, then what is the expected amount of time you will spend in the bank?

4. Consider a post office with two clerks. Three people (Amanda, Ben, and Clark) enter simultaneously. Amanda and Ben elbow Clark out of the way and proceed directly to the clerks. Clark waits until either Amanda and Ben leaves before he begins being served. What is the probability that Amanda is still in the post office after the other two have left, if:
(a) the service time for each clerk is exactly ten minutes (i.e., not random)?
(b) the service times are either 1 minute, or 2 minutes, or 3 minutes, each with probability 1/3?
(c) the service times are exponential with **mean** $1/\mu$?

5. Suppose Machine 1 is currently working. Machine 2 will be put into use at a time t from now. If the lifetime of Machine i (where $i = 1, 2$) is exponential with rate λ_i, then what is the probability that Machine 1 is the first machine to fail? [Hint: Condition on whether Machine 1 fails before time t.]

6. The lifetime of Jon's dog and cat are independent exponential random variables with respective rates λ_d and λ_c. One of the pets has just died. Find the expected *additional* lifetime of the other pet. [Hint: Use conditioning.]

7. A doctor has scheduled two appointments, one at 1:00 p.m. and the other at 1:30 p.m. The amounts of time that the appointments last are independent exponential random variables with mean 30 minutes. Assuming that both patients arrive on time and that the doctor can only meet with one patient at a time, find the expected amount of time that the 1:30 appointment person spends at the doctor's office. [Hint: Use conditioning, and consider a similar type of conditioning as the hint in Problem 5 suggests.]

Graduate Students Must Also Do The Following Problem (Undergraduates may do it for extra credit):

8. Consider a two-server system in which the customer is served first by server 1, then by server 2, and then departs the system. The service times at server i are exponential random variables with **rates** $\mu_i, i = 1, 2$. When you arrive, you find server 1 free and two customers at server 2: customer A being served, and customer B waiting in line.

   (a) Find P_A, the probability that A is still being served when you move over to server 2.

   (b) Find P_B, the probability that B is still in the system when you move over to server 2.

   (c) Find $E[T]$, where T is the time you spend in the system.

   Hint: Write $T = S_1 + S_2 + W_A + W_B$, where S_i is your service time at server i, W_A is the amount of time you wait in line while A is being served, and W_B is the amount of time you wait in line while B is being served.