Day 1 : Problem 1.

PQ

$$X_1, \ldots, X_n \overset{iid}{\sim} f(x|\theta) = \theta^{-c} c x^{c-1} e^{-(x/\theta)^c} I(x>0)$$

(a) The likelihood function is

$$f(\underset{\sim}{X}|\theta) = \theta^{-nc} c^n \prod_{i=1}^n X_i^{c-1} \exp\left(-\frac{1}{\theta^c} \sum_{i=1}^n X_i^c\right)$$

$$l(\theta) = -nc \log\theta + n\log c + (c-1)\sum_{i=1}^n \log X_i - \frac{1}{\theta^c}\sum_{i=1}^n X_i^c$$

Setting $\frac{\partial l}{\partial \theta} = -\frac{nc}{\theta} + \frac{c\sum_{i=1}^n X_i^c}{\theta^{c+1}} = 0$ and solving for θ yields

the MLE for θ. $\hat{\theta}_{MLE} = \left(\frac{\sum_{i=1}^n X_i^c}{n}\right)^{\frac{1}{c}} \triangleq \left(\frac{T(\underset{\sim}{X})}{n}\right)^{\frac{1}{c}}$

$\left(\text{Check second derivative: } \left.\frac{\partial^2 l}{\partial\theta^2}\right|_{\hat\theta_{MLE}} = \frac{-nc^2}{\left(\frac{T}{n}\right)^{\frac{2}{c}}} < 0 \right).$

(b) According to the likelihood in (a), $T(\underset{\sim}{X}) = \sum_{i=1}^n X_i^c$ is a complete sufficient statistic for θ. Moreover, using the pdf technique, one can show that

$X^c \sim \text{Exponential}(\theta^c)$ and thus

$T(X) \sim \text{gamma}(n, \theta^c)$. Therefore

$$E(\hat\theta_{MLE}) = \int_0^\infty \left(\frac{t}{n}\right)^{\frac{1}{c}} \frac{1}{\Gamma(n)\,\theta^{cn}} t^{n-1} e^{-\frac{t}{\theta^c}}\,dt$$

$$= \frac{\Gamma(n+\frac{1}{c})}{\Gamma(n)\, n^{\frac{1}{c}}}\,\theta$$

It follows that the UMVUE for θ is $\frac{\Gamma(n)\,n^{\frac{1}{c}}}{\Gamma(n+\frac{1}{c})}\hat\theta_{MLE}$.

Day 1. Problem 1.

(C) For $\theta_2 > \theta_1 > 0$. the ratio

$$\frac{f(\underset{\sim}{x}, \theta_2)}{f(\underset{\sim}{x}, \theta_1)} = \left(\frac{\theta_1}{\theta_2}\right)^{nc} \exp\left\{\left(\frac{1}{\theta_1^c} - \frac{1}{\theta_2^c}\right) \sum_{i=1}^{n} X_i^c\right\}$$

is an increasing function of $T(\underset{\sim}{X})$. Hence the distribution family of X has an MLR in $T(X)$. By the Karlin-Rubin Theorem, the UMP test of size α is given by

$\delta_0(\underset{\sim}{X}) = I\{T(\underset{\sim}{X}) > t_0\}$, where t_0 satisfies

$P_{\theta_0}\{T(X) > t_0\} = \alpha$, i.e.

$P_{\theta_0}\left\{\frac{2T(X)}{\theta_0^c} > \frac{2t_0}{\theta_0^c}\right\} = \alpha$. note that $\frac{2T(X)}{\theta^c} \sim \chi^2_{2n}$.

or $P_{\theta_0}\left(\chi^2_{2n} > \frac{2t_0}{\theta_0^c}\right) = \alpha$.

Hence, $\frac{2t_0}{\theta_0^c} = \chi^2_{2n,\alpha}$, that is $t_0 = \frac{\theta_0^c \chi^2_{2n,\alpha}}{2}$.

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Day 1. Problem 2.

$X_1, \dots, X_n \overset{iid}{\sim} f(x|\theta) = \theta^{-1} x^{\frac{1-\theta}{\theta}} I(0 \le x \le 1), \quad \theta > 0$

(a) The pdf follows the format of that of an exponential family.

$$f(x|\theta) = I(0 \le x \le 1)\, \theta^{-1} \exp\left(\frac{1-\theta}{\theta} \log x \right)$$

It follows that $\sum_{i=1}^{n} \log X_i$ is a sufficient statistic for θ.
To show that it is also a minimal, consider the ratio

$$\frac{f(\mathbf{X}|\theta)}{f(\mathbf{Y}|\theta)} = \frac{\theta^{-n} I(0 \le X_{(1)} \le X_{(n)} \le 1) \exp\left(\frac{1-\theta}{\theta} \sum_{i=1}^{n} \log X_i \right)}{\theta^{-n} I(0 \le Y_{(1)} \le Y_{(n)} \le 1) \exp\left(\frac{1-\theta}{\theta} \sum_{i=1}^{n} \log Y_i \right)}$$

$$= \frac{I(0 \le X_{(1)} \le X_{(n)} \le 1)}{I(0 \le Y_{(1)} \le Y_{(n)} \le 1)} \exp\left\{ \frac{1-\theta}{\theta} \left(\sum_{i=1}^{n} \log X_i - \sum_{i=1}^{n} \log Y_i \right) \right\},$$

which is free of θ iff $\sum_{i=1}^{n} \log X_i = \sum_{i=1}^{n} \log Y_i$.

Therefore $T(\mathbf{X}) = -2 \sum_{i=1}^{n} \log X_i$ is a minimal sufficient statistic for θ.

(b) Solving $y = -2 \log x$ for x gives $x = e^{-\frac{y}{2}}$.
The pdf of Y is given by

$$f_Y(y|\theta) = \theta^{-1} \left(e^{-\frac{y}{2}} \right)^{\frac{1-\theta}{\theta}} I(y > 0) \cdot \frac{1}{2} e^{-\frac{y}{2}}$$

$$= \frac{1}{2\theta} e^{-\frac{y}{2\theta}} I(y > 0)$$

That is, $Y \sim \text{Exponential}(2\theta)$.

(c) Since $T(\mathbf{X}) = \sum_{i=1}^{n} Y_i$, $T \sim \text{Gamma}(n, 2\theta)$, and thus $\frac{T}{\theta} \sim \chi^2_{2n}$.

It follows that a two-sided 95% confidence interval for θ can be induced from the following.

$$P\left(\chi^2_{2n,0.975} < \frac{T}{\theta} < \chi^2_{2n,0.025}\right) = 0.95 \quad \text{, i.e.,}$$

$$P\left(\frac{T}{\chi^2_{2n,0.025}} < \theta < \frac{T}{\chi^2_{2n,0.975}}\right) = 0.95 .$$

where $\chi^2_{2n,\alpha}$ is a χ^2_{2n} percentile that satisfies $P(\chi^2_{2n} > \chi^2_{2n,\alpha}) = \alpha$.

In summary, a two-sided 95% C.I. is $\left(\frac{T}{\chi^2_{2n,0.025}}, \frac{T}{\chi^2_{2n,0.975}}\right)$

(d) The expected length of the above C.I. is

$$2n\theta\left(\frac{1}{\chi^2_{2n,0.975}} - \frac{1}{\chi^2_{2n,0.025}}\right) \xrightarrow{\text{as } n\to\infty} 0 .$$

One can prove this by approximating χ^2 distribution using normal. That is, $\frac{\chi^2_{2n,0.975}}{n} \approx \frac{2n + 1.96\sqrt{4n}}{n} \xrightarrow{n\to\infty} 2$, and

$$\frac{\chi^2_{2n,0.025}}{n} \approx \frac{2n - 1.96\sqrt{4n}}{n} \xrightarrow{n\to\infty} 2 .$$

Hence $2n\theta\left(\frac{1}{\chi^2_{2n,0.975}} - \frac{1}{\chi^2_{2n,0.025}}\right) \xrightarrow{n\to\infty} 0$

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D1 Prob 3

(1)

(a) First we look at the joint dist of

$$L_1 = c_1 X + c_2 Y \quad \text{and} \quad L_2 = c_3 X + c_4 Y$$

when $X \sim N(\mu_1, \sigma^2)$ and $Y \sim N(\mu_2, \sigma^2)$ w/ $X \perp\!\!\!\perp Y$.
The joint MGF is

$$M_{L_1,L_2}(s,t) = E\left\{ e^{s(c_1X + c_2Y) + t(c_3X + c_4Y)} \right\}$$

$$= E\left\{ e^{(sc_1 + tc_3)X + (sc_2 + tc_4)Y} \right\}$$

$$= M_X(sc_1 + tc_3)\, M_Y(sc_2 + tc_4) \quad \text{by indep.}$$

$$= \exp\left\{ \mu_1(sc_1 + tc_3) + \tfrac{1}{2}(sc_1 + tc_3)^2 \sigma^2 \right\} \times$$

$$\exp\left\{ \mu_2(sc_2 + tc_4) + \tfrac{1}{2}(sc_2 + tc_4)^2 \sigma^2 \right\}$$

$$= \exp\left\{ (\mu_1 c_1 + \mu_2 c_2)s + (\mu_1 c_3 + \mu_2 c_4)t + \tfrac{1}{2}\left[(c_1^2 + c_2^2)s^2 + (c_3^2 + c_4^2)t^2 + 2(c_1c_3 + c_2c_4)st \right] \sigma^2 \right\}$$

This will factor out into a fn of s and t iff

$$c_1 c_3 + c_2 c_4 = 0.$$

This is the condition for independence between L_1 and L_2.

Day 1, Problem 3

Note that in this case,

$$L_1 \sim N\left(\mu_1 c_1 + \mu_2 c_2, (c_1^2 + c_3^2)\sigma^2\right)$$

$$L_2 \sim N\left(\mu_1 c_3 + \mu_4 c_4, (c_3^2 + c_4^2)\sigma^2\right).$$

(b) $V = \frac{X+Y}{\sqrt{2}\sigma} = c_1 X_1 + c_2 Y$ w/ $c_1 = \frac{1}{\sqrt{2}\sigma}$, $c_2 = \frac{1}{\sqrt{2}\sigma}$

$W = \frac{X-Y}{\sqrt{2}\sigma} = c_3 X + c_4 Y$ w/ $c_3 = \frac{1}{\sqrt{2}\sigma}$, $c_4 = -\frac{1}{\sqrt{2}\sigma}$

Since $c_1 c_3 + c_2 c_4 = 0$ then V and W are indep.

(c) When $\mu_1 = \mu_2 = \mu$, then

$$V \sim N\left(\mu\left(\frac{2}{\sqrt{2}\sigma}\right), \left(\frac{1}{2\sigma^2} + \frac{1}{2\sigma^2}\right)\sigma^2\right)$$

$$= N\left(\sqrt{2}\frac{\mu}{\sigma}, 1\right)$$

$$W \sim N\left(\mu(0), \left(\frac{1}{2\sigma^2} + \frac{1}{2\sigma^2}\right)\sigma^2\right)$$

$$= N(0, 1).$$

and $V \perp\!\!\!\perp W$.

(d) When $\mu_1 = \mu_2 = \mu$, we seek dist of $P = \frac{XY}{\sigma^2}$, Several ways to do this.

Given $T = \frac{(X+Y)^2}{2\sigma^2} = t$

Day 1, Problem 3

Observe that

$$P = \frac{XY}{\sigma^2} = \left\{ \left(\frac{X+Y}{\sigma}\right)^2 - \left(\frac{X-Y}{\sigma}\right)^2 \right\} \frac{1}{4}$$

$$= \left\{ 2V^2 - 2W^2 \right\} \frac{1}{4}$$

$$= \frac{1}{2}\left(V^2 - W^2\right).$$

Since we want $(P \mid T = t) = (P \mid |V| = \sqrt{t})$, and noting that W^2 and $|V|$ are indep for previous result, and also the fact that

$$W^2 \sim \chi^2_1 = \text{chi-square with 1 d.f.}$$

then

$$P \mid T = t \sim \frac{1}{2}\left(t - \chi^2_1\right).$$

Note: Other approaches would be using the transformation theorem.