Day 2, January 2011.

Prob 1 (Solution).

(a). By Central Limit Theorem, one has

$$\sqrt{n}\left(\bar{X}-\lambda\right) \xrightarrow{d} N(0,\lambda).$$

Let $g(\lambda)=e^{-\lambda}$. Using delta method, one has

$$\sqrt{n}\left(g(\bar{X})-g(\lambda)\right) \xrightarrow{d} N\left(0,\ (g'(\lambda))^2\cdot\lambda\right)$$

i.e., $\sqrt{n}\left(\underbrace{e^{-\bar{X}}}_{\text{This is } T_{n,1}}-e^{-\lambda}\right) \xrightarrow{d} N\left(0,\ \lambda e^{-2\lambda}\right)$

(b) Let $Y_i=\mathbb{1}_{(X_i=0)}$ for $i=1,\cdots,n$.

Then Y_i's are iid r.vs and $Y_i \sim \text{Bernoulli}(p=e^{-\lambda})$.

$E(Y_1)=p=e^{-\lambda}$. $E(Y_1^2)=E(Y_1)=e^{-\lambda}$.

$Var(Y_1)=e^{-\lambda}-(e^{-\lambda})^2=e^{-\lambda}(1-e^{-\lambda})$.

By using Central Limit Theorem, one has

$\bar{Y} \dot{\sim} N\left(p,\ \frac{p(1-p)}{n}\right)$ or $\sqrt{n}\left(\bar{Y}-p\right) \xrightarrow{d} N\left(0,\ p(1-p)\right)$

i.e., $\sqrt{n}\left(\underbrace{\frac{1}{n}\sum_i \mathbb{1}_{(X_i=0)}}_{\text{This is } T_{n,2}}-e^{-\lambda}\right) \xrightarrow{d} N\left(0,\ e^{-\lambda}(1-e^{-\lambda})\right)$

Day 2, Problem 1 Solution (continued)

(c) Consider the ratio of the asymptotic variances of $T_{n,1}$ and $T_{n,2}$,

$$\frac{\lambda e^{-2\lambda}}{e^{-\lambda}\cdot(1-e^{-\lambda})} = \frac{\lambda}{e^{\lambda}-1} < 1$$

$\boxed{\text{Since } \lambda > 0, \quad e^{\lambda}-1 > \lambda \text{ always holds.}}$

Thus, $T_{n,1}$ has a smaller asymptotic variance. Notice that both estimators are asymptotically unbiased, so $T_{n,1}$ is more efficient than $T_{n,2}$ in estimating $e^{-\lambda}$ when sample size is large.

Day 2 Prob 2 $\quad X_1, X_2, \ldots, X_n \overset{iid}{\sim} f(x|\lambda) = \lambda e^{-\lambda x}, \; x > 0.$

$X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ are the o.s.

$$D_i = (n-i+1)(X_{(i)} - X_{(i-1)}), \quad \cancel{X}_{(0)} = 0.$$

(a) Joint Dist of the D_i's.

First the jpdf of the $X_{(1)}, \ldots, X_{(n)}$ is

$$f(x_{(1)}, \ldots, x_{(n)}) = n! \prod_{i=1}^{n} \lambda e^{-\lambda x_{(i)}}, \quad 0 < x_{(1)} < \cdots < x_{(n)} < \infty.$$

$$= n! \lambda^n e^{-\lambda \sum_1^n x_{(i)}}.$$

Observe that

$$X_{(1)} = \frac{D_1}{n}$$

$$X_{(2)} = \frac{D_2}{n-1} + X_{(1)} = \frac{D_1}{n} + \frac{D_2}{n-1}$$

$$X_{(3)} = \frac{D_3}{n-2} + X_{(2)} = \frac{D_1}{n} + \frac{D_2}{n-1} + \frac{D_3}{n-2}$$

$$\vdots$$

$$X_{(k)} = \sum_{j=1}^{k} \frac{D_j}{n-j+1}$$

$$\vdots$$

$$X_{(n)} = \sum_{j=1}^{n} \frac{D_j}{n-j+1}.$$

Also, note that $D_i \geq 0 \ \forall i$. The Jacobian of transformation is

$$|J| = \begin{vmatrix} \frac{1}{n} & 0 & \longrightarrow & & 0 \\ \frac{1}{n} & \frac{1}{n-1} & 0 & \longrightarrow & 0 \\ \vdots & & & & \\ \frac{1}{n} & \frac{1}{n-1} & \frac{1}{n-2} & \longrightarrow & 1 \end{vmatrix} = \frac{1}{n!}$$

Therefore,

$$f_{D_1,\dots,D_n}(d_1,\dots,d_n) = n!\,\lambda^n \cancel{\prod_{i=1}^{n}}\, e^{-\lambda \sum_{i=1}^{n}\left(\sum_{j=1}^{i} \frac{d_j}{n-j+1}\right)} \cdot \frac{1}{n!}, \quad d_i \geq 0$$

$$= \lambda^n \exp\left\{-\lambda \sum_{j=1}^{n}\left(\sum_{i=j}^{n} \frac{d_j}{n-j+1}\right)\right\}, \quad d_j \geq 0$$

$$= \lambda^n \exp\left\{-\lambda \sum_{j=1}^{n} (n-j+1)\frac{d_j}{n-j+1}\right\}.$$

$$= \lambda^n \exp\left\{-\lambda \sum_{j=1}^{n} d_j\right\}, \quad d_j \geq 0$$

$\therefore$

$$D_1, D_2, \dots, D_n \overset{iid}{\sim} \text{Exp}(\lambda).$$

Day 2, Problem 2

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(b) Since $X_{(n)} = \sum_{j=1}^{n} \frac{D_j}{n-j+1}$, then

$$\mathbb{E}(X_{(n)}) = \frac{1}{\lambda} \sum_{j=1}^{n} \frac{1}{n-j+1}.$$

(c) The TTOT statistic based on $X_{(1)}, \dots, X_{(k)}$ and $X_{(j)} \geq X_{(k)}$ for $j > k$ is

$$T = \sum_{j=1}^{k} X_{(j)} + (n-k) X_{(k)}.$$

$$= \sum_{j=1}^{k} \left(\sum_{i=1}^{j} \frac{D_i}{n-i+1} \right) + (n-k) \sum_{i=1}^{k} \frac{D_i}{n-i+1}$$

$$= \sum_{i=1}^{k} \sum_{j=i}^{k} \frac{D_i}{n-i+1} + (n-k) \sum_{i=1}^{k} \frac{D_i}{n-i+1}$$

$$= \sum_{i=1}^{k} \frac{D_i}{n-i+1} \cdot (k-i+1) + (n-k) \sum_{i=1}^{k} \frac{D_i}{n-i+1}$$

$$= \sum_{i=1}^{k} \frac{D_i}{n-i+1} \left[k-i+1+n-k \right].$$

$$= \sum_{i=1}^{k} D_i .$$

Day 2, Problem 2

Now, since $D_i \overset{iid}{\sim} EXP(\lambda)$, then

$T \sim Ga(K, \lambda)$. We know $E(T) = \frac{K}{\lambda}$.

Consider an estimator of form

$$\delta_c = \frac{c}{T}. \quad \text{for } c \text{ constant.}$$

Then

$$E(\delta_c) = c\, E\left(\frac{1}{T}\right)$$

$$= c \int_0^\infty \frac{1}{t} \cdot \frac{\lambda^K}{\Gamma(K)} t^{K-1} e^{-\lambda t}\, dt$$

$$= c \frac{\lambda^K}{\Gamma(K)} \int_0^\infty t^{(K-1)-1} e^{-\lambda t}\, dt$$

$$= c \frac{\lambda^K}{\Gamma(K)} \frac{\Gamma(K-1)}{\lambda^{K-1}}$$

$$= c \frac{\lambda}{K-1}.$$

Taking $c = K-1$, we find that

$\delta_{K-1} = \frac{K-1}{T}$ is unbiased for λ, for $K > 1$.

Day 2, Problem 2

(d) Variance of $\delta_{K-1} = \frac{k-1}{T}$. We have

$$E\left(\left(\frac{k-1}{T}\right)^2\right) = (k-1)^2 E\left(\frac{1}{T^2}\right)$$

$$= (k-1)^2 \int_0^\infty \frac{1}{t^2} \frac{\lambda^k}{\Gamma(k)} t^{k-1} e^{-\lambda t} dt$$

$$= (k-1)^2 \frac{\lambda^k}{\Gamma(k)} \int_0^\infty t^{(k-2)-1} e^{-\lambda t} dt$$

$$= (k-1)^2 \frac{\lambda^k}{\Gamma(k)} \frac{\Gamma(k-2)}{\lambda^{k-2}}$$

$$= \frac{(k-1)^2 \lambda^2}{(k-1)(k-2)} = \left(\frac{k-1}{k-2}\right)\lambda^2.$$

$\therefore$

since unbiased

$$Var\left(\frac{k-1}{T}\right) = \left(\frac{k-1}{k-2}\right)\lambda^2 - \lambda^2$$

$$= \lambda^2 \left\{\frac{(k-1)-(k-2)}{k-2}\right\}$$

$$= \frac{\lambda^2}{k-2}, \quad k > 2.$$

Day 2, Prob 3

$X \sim POI(\mu)$, $Y \sim POI(\nu)$, $X \perp\!\!\!\perp Y$.

(a) $P(X=x \mid S = X+Y = s)$

$$= \frac{P(X=x, Y=s-x)}{P(S=s)}$$

if $s<0$, prob is 0.

$$= \frac{P(X=x)P(Y=s-x)}{P(S=s)}$$

$$P(S=s) = \sum_{x=0}^{s} P(X=x)P(Y=s-x)$$

$$= \sum_{x=0}^{s} \frac{e^{-\mu}\mu^{x}}{x!}\frac{e^{-\nu}\nu^{s-x}}{(s-x)!}$$

$$= e^{-(\mu+\nu)}\frac{1}{s!}\sum_{x=0}^{s}\binom{s}{x}\mu^{x}\nu^{s-x}$$

$$= \frac{e^{-(\mu+\nu)}(\mu+\nu)^{s}}{s!}$$

so $S \sim POI(\mu+\nu)$.

Day 2, Problem 3

$$P(X=x|S=s) = \frac{\frac{e^{-\mu}\mu^x}{x!}\frac{e^{-\nu}\nu^{s-x}}{(s-x)!}}{\frac{e^{-(\mu+\nu)}(\mu+\nu)^s}{s!}}$$

$$= \binom{s}{x}\left(\frac{\mu}{\mu+\nu}\right)^x\left(1-\frac{\mu}{\mu+\nu}\right)^{s-x}$$

$$= \binom{s}{x} p^x (1-p)^{s-x}, \quad x=0,1,\dots,s.$$

$\therefore$

$$X|S=s \sim BIN\left(s, p=\frac{\mu}{\mu+\nu}\right).$$

(b) Under $H_0: \mu=\nu, \nu \in \mathbb{R}_+$,

$$X|S=s \sim BIN\left(s, p=\tfrac{1}{2}\right)$$

Under $H_1: \mu=2\nu, \nu \in \mathbb{R}_+$

$$X|S=s \sim BIN\left(s, p=\frac{2}{3}\right). \quad \frac{\mu}{\mu+\nu} = \frac{2\nu}{2\nu+\nu} = \frac{2}{3}$$

(c) Given $S=s$, the MP ~~a~~ test of H_0 vs H_1 would be of form and

$$\gamma_S = \frac{\alpha - B(S, k_S - 1)}{b(S, k_S)}$$

$$= \frac{\alpha - \sum_{j=1}^{k_S - 1} \binom{S}{j} \left(\frac{1}{2}\right)^S}{\binom{S}{k_S} \left(\frac{1}{2}\right)^S}.$$

(d) Power of test in (c).

$$\pi(\mu, \nu) = E\left(\delta_S(X) \,\middle|\, \substack{\mu, \nu \\ \mu = 2\nu}\right)$$

$$= E\left\{ E\left[\delta_S(X) \,\middle|\, S, \substack{\mu, \nu \\ \mu = 2\nu}\right]\right\}$$

$$= E\left\{ P\left(X < k_S \,\middle|\, p = \tfrac{2}{3}, S\right) + \gamma_S P\left(X = k_S \,\middle|\, p = \tfrac{2}{3}, S\right)\right\}$$

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$$= \sum_{s=0}^{\infty} \frac{e^{-(\mu+\nu)}(\mu+\nu)^s}{s!} \left\{ \sum_{j=0}^{k_s - 1} \binom{s}{j} \left(\frac{2}{3}\right)^j \left(\frac{1}{3}\right)^{s-j} + \gamma_s \binom{s}{k_s} \left(\frac{2}{3}\right)^{k_s} \left(\frac{1}{3}\right)^{s-k_s} \right\}$$

where the k_s and γ_s are as defined in part (c).

Day 2, Problem 3

(3)

$$\delta_s(x) = \begin{cases} 1 & \text{if } R_s(x) > k_s \\ \gamma_s & \quad = \\ 0 & \quad < \end{cases}$$

where

$$R_s(x) = \frac{\binom{s}{x}\left(\frac{2}{3}\right)^x \left(\frac{1}{3}\right)^{s-x}}{\binom{s}{x}\left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{s-x}}$$

$$\frac{\left(\frac{2}{3}\right)^x \left(\frac{1}{3}\right)^{s-x}}{\left(\frac{1}{2}\right)^s}$$

$$= \left(\frac{2}{3}\right)^x \left(\frac{2}{3}\right)^{s-x} 2^{s-x}$$

$$= 2^x \cdot 3^{-x} \cdot 3^{x-s} \cdot 2^s$$

$$= 2^{s+x}\, 3^{-s}$$

$$= \left(\frac{2}{3}\right)^s 2^{s-x} \,.$$

$$= \left(\frac{2}{3}\right)^s \cdot 2^x$$

Since this is increasing in x,

$$\delta_s(x) = \begin{cases} 1 & \text{if } x > k_s \\ \gamma_s & \quad = \\ 0 & \quad < \end{cases}$$

where we choose

cdf of $\mathrm{Bin}(s, \frac{1}{2})$.

$$k_s = \max\left\{m : B(s, m-1) \geq 1-\alpha\right\}$$

$$= \max\left\{m : \sum_{j=0}^{m} \binom{s}{j}\left(\frac{1}{2}\right)^s \geq 1-\alpha\right\}$$