Formula Sheet

May 2011 PhD Qualifying Examination

- Beta(α, β):

  pdf: $f(x|\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad 0 \le x \le 1,\ \alpha > 0,\ \beta > 0.$

  $E(X) = \frac{\alpha}{\alpha+\beta}$, $\mathrm{Var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$.

- Cauchy(θ, σ):

  pdf: $f(x|\theta,\sigma) = \frac{1}{\pi\sigma} \frac{1}{1+\left(\frac{x-\theta}{\sigma}\right)^2}, \quad -\infty < x < \infty,\ -\infty < \theta < \infty,\ \sigma > 0.$

- Laplace(μ, σ):

  pdf: $f(x|\mu,\sigma) = \frac{1}{2\sigma} e^{-|x-\mu|/\sigma}, \quad -\infty < x < \infty,\ -\infty < \mu < \infty,\ \sigma > 0.$

  $E(X) = \mu$, $\mathrm{Var}(X) = 2\sigma^2$.

  MGF: $M_X(t) = \frac{e^{\mu t}}{1-\sigma^2 t^2}$, $|t| < 1/\sigma$.

- Gamma$(\alpha,\ \beta)$:

  pdf: $f(x|\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}$, $x \ge 0,\ \alpha, \beta > 0.$

  $E(X) = \alpha\beta$, $\mathrm{Var}(X) = \alpha\beta^2$.

  MGF: $M_X(t) = \left(\frac{1}{1-\beta t}\right)^\alpha$, $t < 1/\beta$.

- Exponential(β) is a special case of gamma(α, β) with $\alpha = 1$.

- Chi square(p) is a special case of gamma distribution with $\alpha = p/2$ and $\beta = 2$.

- F distribution $F(\nu_1, \nu_2)$:

  pdf: $f(x|\nu_1,\nu_2) = \frac{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)} \left(\frac{\nu_1}{\nu_2}\right)^{\nu_1/2} \frac{x^{(\nu_1-2)/2}}{\left(1+\frac{\nu_1}{\nu_2}x\right)^{(\nu_1+\nu_2)/2}}, \quad x \ge 0,\ \nu_1,\ \nu_2 = 1, 2, \ldots.$

  $E(X) = \frac{\nu_2}{\nu_2-2}$, $\nu_2 > 2$, $\mathrm{Var}(X) = 2\left(\frac{\nu_2}{\nu_2-2}\right)^2 \frac{\nu_1+\nu_2-2}{\nu_1(\nu_2-4)}$, $\nu_2 > 4$.

- Pareto(α, β):

  pdf: $f(x|\alpha,\beta) = \frac{\beta\alpha^\beta}{x^{\beta+1}}, \quad x > \alpha,\ \alpha > 0,\ \beta > 0.$

  $E(X) = \frac{\beta\alpha}{\beta-1}$, $\beta > 1$, $\mathrm{Var}(X) = \frac{\beta\alpha^2}{(\beta-1)^2(\beta-2)}$, $\beta > 2$.

- Uniform(a, b):

  pdf: $f(x|a,b) = \frac{1}{b-a}, \quad a \le x \le b.$

$E(X) = \frac{a+b}{2}$, $\text{Var}(X) = \frac{(b-a)^2}{12}$.

MGF: $M_X(t) = \frac{e^{bt}-e^{at}}{(b-a)^t}$.

- Lognormal(μ, σ^2):

  pdf: $f(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} x^{-1} \exp\{-(\ln x - \mu)^2/2\sigma^2\}, \quad x > 0,\ -\infty < \mu < \infty,\ \sigma^2 > 0.$

  $E(X) = \exp\{\mu + \sigma^2/2\}$, $\text{Var}(X) = \exp\{2(\mu + \sigma^2)\} - \exp\{2\mu + \sigma^2\}$.

- t distribution with ν degrees of freedom:

  pdf: $f(x|\nu) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \frac{1}{\sqrt{\nu\pi}} \left(1 + \frac{x^2}{\nu}\right)^{-(\nu+1)/2}, \quad \nu = 1, 2, \ldots.$

  $E(X) = 0$, $\nu > 1$, $\text{Var}(X) = \frac{\nu}{\nu-2}$, $\nu > 2$.

- Weibull(γ, β):

  pdf: $f(x|\gamma, \beta) = \frac{\gamma}{\beta} x^{\gamma-1} e^{-x^\gamma/\beta}, \quad x \geq 0,\ \gamma > 0,\ \beta > 0.$

  $E(X) = \beta^{1/\gamma}\Gamma(1 + 1/\gamma)$, $\text{Var}(X) = \beta^{2/\gamma}\left\{\Gamma(1 + 2/\gamma) - \Gamma^2(1 + 1/\gamma)\right\}$.

- Binomial(n, p):

  pmf: $P(X = x|n, p) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, 2, \ldots, n;\ 0 \leq p \leq 1.$

  $E(X) = np$, $\text{Var}(X) = np(1-p)$.

  MGF: $M_X(t) = (pe^t + 1 - p)^n$.

- Discrete uniform:

  pmf: $P(X = x|N) = \frac{1}{N}, \quad x = 1, 2, \ldots, N;\ N = 1, 2, \ldots.$

  $E(X) = \frac{N+1}{2}$, $\text{Var}(X) = \frac{(N+1)(N-1)}{12}$.

  MGF: $M_X(t) = \frac{1}{N}\sum_{i=1}^N e^{it}$.

- Geometric(p):

  pmf: $P(X = x|p) = p(1-p)^{x-1}, \quad x = 1, 2, \ldots, 0 \leq p \leq 1.$

  $E(X) = \frac{1}{p}$, $\text{Var}(X) = \frac{1-p}{p^2}$.

  MGF: $M_X(t) = \frac{pe^t}{1-(1-p)e^t}$, $t < -\log(1-p)$.

- Poisson(λ):

  pmf:

$$P(X = x|\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, \ldots;\ \lambda \geq 0.$$

$E(X) = \lambda$, $\text{Var}(X) = \lambda$.

MGF: $M_X(t) = e^{\lambda(e^t - 1)}$.

- About gamma function $\Gamma(\alpha)$:

  For $\alpha > 0$,

$$
\begin{aligned}
\Gamma(\alpha) &= \int_0^\infty t^{\alpha-1} e^{-t} dt, \\
\Gamma(\alpha+1) &= \alpha\Gamma(\alpha), \\
\Gamma(n) &= (n-1)!, \quad \text{for any integer } n > 0, \\
\Gamma\left(\frac{1}{2}\right) &= \sqrt{\pi}, \\
\psi(\alpha) &\triangleq \frac{d}{d\alpha} \log \Gamma(\alpha) = \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} \quad \text{(digamma function)}, \\
\psi_1(\alpha) &\triangleq \frac{d^2}{d\alpha^2} \log \Gamma(\alpha) \quad \text{(trigamma function)}.
\end{aligned}
$$

- $\sum_{i=0}^{n} t^i = \frac{1 - t^{n+1}}{1 - t}$, for $t \neq 1$.

- $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

- $\sum_{i=0}^{+\infty} \frac{x^i}{i!} = e^x$.