DAY 2, Q1

1LD3

$$f_{X,Y}(x,y) = \frac{1}{(2\pi)}\frac{1}{\sqrt{xy}} e^{-\frac{(x+y)}{2}}, \quad x>0, \; y>0$$

Let $U = X+Y$ $\quad V = \frac{X}{Y}$

$$\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial u}{\partial y} = 1$$

$$\frac{\partial v}{\partial x} = \frac{1}{y} \qquad \frac{\partial v}{\partial y} = -\frac{x}{y^2}$$

$$|J|^{-1} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ \frac{1}{y} & -\frac{x}{y^2} \end{vmatrix} = \left| -\frac{x}{y^2} - \frac{1}{y} \right|$$

$$= \left| \frac{x+y}{y^2} \right|$$

$$\Rightarrow |J| = \left| \frac{y^2}{x+y} \right|$$

Hence,

$$f_{U,V}(u,v) = f_{X,Y}(x(u,v), y(u,v)) \left| \frac{y^2}{x+y} \right|$$

$$= \frac{1}{(2\pi)}\frac{y^2}{\sqrt{xy}\,(x+y)} e^{-(x+y)/2}, \quad x,y>0$$

$$= \frac{1}{2\pi}\frac{y^{3/2}}{\sqrt{x}}\frac{1}{u} e^{-u/2}, \quad u,v>0$$

$$Y = \frac{X}{V} \qquad X = U - Y$$

$$\Rightarrow \quad Y = \frac{U-Y}{V} \quad \Rightarrow Y\left\{1 + \frac{1}{V}\right\} = U$$

$$\Rightarrow Y = \frac{U}{V+1}$$

$$X = \frac{U(V+1)-U}{V+1} = \frac{UV}{V+1}.$$

So,

$$f_{U,V}(u,v) = \frac{u^{3/2} \quad (v+1)^{1/2}}{2\pi (v+1)^{3/2} \; u \sqrt{uv}} \, e^{-u/2}$$

$$= \frac{1}{2\pi \sqrt{v} \, (v+1)} \, e^{-u/2} \qquad , \; u, v > 0.$$

---

b)

$$f_V(v) = \int_0^\infty \frac{1}{v^{\frac{1}{2}} (v+1) 2\pi} \, e^{-u/2} \, du$$

But $\displaystyle\int_0^\infty e^{-u/2} \, du = \left[-2e^{-u/2}\right]_0^\infty = 2.$

So

$$f_V(v) = \frac{1}{v^{1/2} (v+1) \pi} \qquad , \; v > 0.$$

c) $f_U(u) = \int_0^\infty \frac{1}{v^{\frac{1}{2}}(v+1)\pi 2} e^{-u/2} \, dv$

$= \frac{1}{2} e^{-u/2}$ , $u > 0$.

because $\int_0^\infty f_V(v)\,dv = 1$

So $U \sim \chi^2_2 \equiv \exp(\lambda = \frac{1}{2})$

d) Yes, U, V are independent because

$f_{UV}(u,v) = f_U(u) f_V(v)$ , $u, v > 0$.

e) $Z = \frac{1}{k}\sum_{i=1}^{k} U_i \approx N\left(E(Z), Var(Z)\right)$ using the Central Limit Theorem.

for $\exp(\lambda = 1/2)$ : $E(X) = \frac{1}{\lambda}$, $Var(X) = 1/\lambda^2$

So, $E(U_i) = \frac{1}{\lambda} = 2$, $Var(U_i) = \lambda^{-2} = 4$

then

$E(Z) = \frac{k2}{k} = 2$ , $Var(Z) = \frac{k4}{k^2} = \frac{4}{k}$

Hence,

$Z \approx N\left(2, \frac{4}{k}\right)$.

f).

$$H_0: f(x) = f_0(x)$$
$$H_1: f(x) = f_1(x)$$

Using the Neyman-Pearson Lemma the most powerful critical region (CR) for testing $H_0$ vs $H_1$ is

$$\lambda = \frac{\prod_{i=1}^{n} f_0(x_i)}{\prod_{i=1}^{n} f_1(x_i)} < \lambda^* .$$

Now,
$$\lambda = \frac{\frac{1}{(2\pi)^{n/2}} \sqrt{\frac{1}{\prod_{i=1}^{n} x_i}}\, e^{-\sum_{i=1}^{n} x_i/2}}{\frac{(\sqrt{2})^n}{\pi^{n/2}}\, e^{-\sum_{i=1}^{n} x_i^2/2}}$$

$$= \left(\frac{1}{2}\right)^n \left(\prod_{i=1}^{n} x_i\right)^{-1/2} e^{\frac{1}{2}\left(\sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} x_i\right)}$$

The CR equivalent to

$$-\frac{1}{2}\sum_{i=1}^{n} \log x_i + \frac{1}{2}\sum_{i=1}^{n}\left(x_i - \frac{1}{2}\right)^2 < c .$$

ie

$$\sum_{i=1}^{n} \log x_i + \sum_{i=1}^{n}\left(x_i - \frac{1}{2}\right)^2 < k$$

DAY 2, QU2.



(HXZ-2) $X_i \overset{iid}{\sim} N(\mu, \sigma^2)$, $i=1,\cdots,n$.
$\sigma^2>0$ known

(a) $$\text{CRLB} = \frac{e^{2t\mu}\, t^2}{I(\mu)_n}$$

where $I(\mu)$ is the Fisher information.

As $f(x|\mu) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$.

$$\frac{\partial \log f(x|\mu)}{\partial \mu} = \frac{x-\mu}{\sigma^2},$$

One has $I(\mu) = nE\left[\left\{\frac{\partial \log f(X|\mu)}{\partial\mu}\right\}^2\right] = \frac{n}{\sigma^4} E\{(X-\mu)^2\} = \frac{n}{\sigma^2}$

Therefore, $\text{CRLB} = \sigma^2 t^2 e^{2t\mu}/n$.

(b) $\bar{X}$ is a complete sufficient statistic for $\mu$.
Note that $E(e^{t\bar{X}}) = e^{\mu t + \sigma^2 t^2/(2n)}$, thus
$$E(e^{t\bar{X} - \sigma^2 t^2/(2n)}) = e^{\mu t}.$$
Therefore the UMVUE for $\tau(\mu) = e^{t\mu}$ is $\hat{\tau} = e^{t\bar{X} - \sigma^2 t^2/(2n)}$.
(Lehmann-Scheffé Theorem)

(c)
$$\begin{aligned}
Var(\hat{\tau}) &= e^{-\sigma^2 t^2/n}\, Var(e^{t\bar{X}}) \\
&= e^{-\sigma^2 t^2/n}\left(e^{2\mu t + 2\sigma^2 t^2/n} - e^{2\mu t + \frac{\sigma^2 t^2}{n}}\right) \\
&= e^{2\mu t}\left(e^{\frac{\sigma^2 t^2}{n}} - 1\right) \\
&\neq \text{CRLB}.
\end{aligned}$$

Since $Var(\hat{\tau}) \geq \text{CRLB}$ (by the definition of "LB"),
$Var(\hat{\tau}) > \text{CRLB}$.

$$\frac{Var(\hat{\tau})}{CRLB} = \frac{n(e^{\frac{b^2t^2}{n}}-1)}{b^2t^2} \xrightarrow{n\to\infty} 1$$

because $\lim_{x\to 0} \frac{e^x-1}{x} = 1$. (view $x = \frac{b^2t^2}{n}$).

DAY 2, Q3 ·

ter (more "parallel").

Controlling the FER ($= 0.05$), and adjusting for the chimp, there is a significant difference in mean time it takes chimps to learn the words. Listen takes significantly less time to learn than food, fruit, hat, look, and string. Drink and show take less time than look and string. Key and more take less time than string. A clever student could quantify this using 95% confidence intervals for the difference in log-time, and exponentiate to give 95% confidence intervals for factors of how median time-to-learning the words is different.

2. *The shifted Exponential*: Let $X_1, \ldots, X_n$ be a random sample from the density

$$f(x|\theta, \alpha) = \frac{1}{\theta} \exp\left(-\frac{x-\alpha}{\theta}\right) I_{[\alpha,\infty)}(x),$$

for $\theta > 0$ and real $\alpha$. Suppose $\theta$ and $\alpha$ are unknown.

   (a) Let the first order statistic be denoted $X_{(1)} = \min\{X_1, \ldots, X_n\}$. Find the distribution of
   $$W = \frac{n(X_{(1)} - \alpha)}{\theta}.$$

   $\boxed{\text{Answer}}$

   For $x < \alpha$, $P(X_i > x) = 0$. For $x \geq \alpha$,

$$\begin{aligned} P(X_i > x) &= \int_x^\infty \theta^{-1} e^{-(s-\alpha)/\theta} ds \\ &= \left[-e^{-(s-\alpha)/\theta}\right]_x^\infty \\ &= 0 - \left(-e^{-(x-\alpha)/\theta}\right) \\ &= e^{-(x-\alpha)/\theta}. \end{aligned}$$

$$
\begin{aligned}
P\{n(X_{(1)} - \alpha)/\theta > x\} &= P\{X_{(1)} > x\theta/n + \alpha\} \\
&= P\{X_1 > x\theta/n + \alpha, \ldots, X_n > x/n + \alpha\} \\
&\overset{ind}{=} \prod_{i=1}^{n} P\{X_i > x\theta/n + \alpha\} \\
&= P\{X_1 > x\theta/n + \alpha\}^n \\
&= \left[e^{-(x\theta/n+\alpha-\alpha)/\theta}\right]^n \\
&= \left[e^{-(x\theta/n)/\theta}\right]^n \\
&= e^{-x}.
\end{aligned}
$$

So $W \sim \exp(1)$.

(b) Derive the MLE $(\hat{\theta}, \hat{\alpha})$ for $(\theta, \alpha)$. Hint: First find the MLE $\hat{\alpha}$, then use $\hat{\alpha}$ to find the MLE $\hat{\theta}$.

$\boxed{\text{Answer}}$ The likelihood is

$$
\mathcal{L}(\theta, \alpha) = \prod_{i=1}^{n} \theta^{-1} \exp\left(-\frac{x_i - \alpha}{\theta}\right) I_{[\alpha,\infty)}(x_i).
$$

It's easy to verify that this is zero for $\alpha > x_{(1)}$, where $x_{(1)} = \min\{x_1, \ldots, x_n\}$. Also, $\frac{\partial}{\partial \alpha}\mathcal{L}(\theta, \alpha) > 0$ for $\alpha \le x_{(1)}$. So $\hat{\alpha} = x_{(1)}$. Given $\hat{\alpha}$, the range of $\theta$ does not depend on the data and so we can go the usual route involving the dervative of the log-likelihood function

$$
l(\theta, \hat{\alpha}) = -n \log \theta - \theta^{-1} \sum_{i=1}^{n} (x_i - x_{(1)}) = -n \log \theta - \theta^{-1} n\bar{x} + \theta^{-1} n x_{(1)}.
$$

$$
\frac{d}{d\theta} l(\theta, \hat{\alpha}) = -\frac{n}{\theta} + \frac{n\bar{x}}{\theta^2} - \frac{n x_{(1)}}{\theta^2} \overset{set}{=} 0,
$$

yielding $\hat{\theta} = \bar{x} - x_{(1)}$. A dedicated student will go the extra mile and show $\frac{d^2}{d\theta^2} l(\theta, \hat{\alpha}) < 0$.

(c) Use (a) and (b) to develop a 95% confidence interval for $\alpha$, by plugging in $\hat{\theta}$ for $\theta$. The times it took for $n = 10$ preschoolers to complete a task in minutes are

2.3 1.6 1.1 1.7 1.1 1.7 1.2 1.6 4.4 2.4.

Find, *and interpret*, a 95% confidence interval for $\alpha$ for these data.

$\boxed{\text{Answer}}$ There are a few ways to do this; here's one.

Let $W \sim exp(1)$ and $P(a < W < b) = 0.95$. Solving gives

$$P\left\{X_{(1)} - b\theta/n + < \alpha < X_{(1)} - a\theta/n+\right\} = 0.95.$$

The sample statistics are $\bar{x} = 1.91$ and $x_{(1)} = 1.1$, so $\hat{\theta} = 1.91 - 1.1 = 0.81$. Giving 0.025 to each "tail" yields $P(0.025 < W < 3.69) = 0.95$, and a 95% CI $(0.801, 1.098)$. We are 95% "confident" that the *shortest* amount of time to possibly complete the task is between 0.8 and 1.1 minutes.

(d) Derive the method of moment estimators $(\tilde{\theta}, \tilde{\alpha})$ for $(\theta, \alpha)$.

$\boxed{\text{Answer}}$

$$\mu_1 = E(X_i) = \alpha + \theta.$$

$$\mu_2 = E(X_i^2) = var(X_i) + \mu_1^2 = \theta^2 + \alpha^2 + 2\alpha\theta + \theta^2 = \alpha^2 + 2\alpha\theta + 2\theta^2.$$

Let

$$m_1 = n^{-1}\sum_{i=1}^{n} X_i \text{ and } m_2 = n^{-1}\sum_{i=1}^{n} X_i^2.$$

Then we solve

$$\begin{aligned} m_1 &= \alpha + \theta \\ m_2 &= \alpha^2 + 2\alpha\theta + 2\theta^2 \end{aligned}$$

Taking $\alpha = m_1 - \theta$ (first equation) and plugging into the second equation, we have

$$m_2 = (m_1 - \theta)^2 + 2(m_1 - \theta)\theta + 2\theta^2 = m_1^2 - 2m_1\theta + \theta^2 + 2m_1\theta - 2\theta^2 + 2\theta^2,$$

yielding the quadratic in $\theta$

$$\theta^2 + m_1^2 - m_2 = 0.$$

The quadratic formula gives

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2(0) \pm \sqrt{4(0)^2 - 4(1)(m_1^2 - m_2)}}{2(1)} = \pm\sqrt{m_2 - m_1^2}.$$

So $\tilde{\theta} = \sqrt{m_2 - m_1^2}$ whenever $m_2 > m_1^2$. The first equation then gives $\tilde{\alpha} = m_1 - \sqrt{m_2 - m_1^2}$. Here we require $m_1^2 > m_2 - m_1^2$. Both inequalities yield $m_1^2 < m_2 < 2m_1^2$.

or. $E(X) = \alpha + \theta$, $var(x) \overset{7}{=} \theta^2$

So $\tilde{\alpha} = \bar{x} - s$, provided $\bar{x} > s$.

$\tilde{\theta} = s$