Random Variable: A variable whose value is the numerical outcome of an experiment or random phenomenon.

Discrete Random Variable: A numerical r.v. that takes on a countable number of values (there are gaps in the range of possible values).

Examples:
1. Number of phone calls received in a day by a company
2. Number of heads in 5 tosses of a coin

Continuous Random Variable: A numerical r.v. that takes on an uncountable number of values (possible values lie in an unbroken interval).

Examples:
1. Length of nails produced at a factory
2. Time in 100-meter dash for runners

Other examples?

The probability distribution of a random variable is a graph, table, or formula which tells what values the r.v. can take and the probability that it takes each of those values.
Example 1: Roll 1 die. The r.v. $X = \text{number of dots showing.}$

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(x)$</td>
<td>$\frac{1}{6}$</td>
<td>$\frac{1}{6}$</td>
<td>$\frac{1}{6}$</td>
<td>$\frac{1}{6}$</td>
<td>$\frac{1}{6}$</td>
<td>$\frac{1}{6}$</td>
</tr>
</tbody>
</table>

Example 2: Toss 2 coins. The r.v. $X = \text{number of heads showing.}$

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(x)$</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{4}$</td>
</tr>
</tbody>
</table>

Graph for Example 2:

For any probability distribution:

1. $P(x)$ is between 0 and 1 for any value of $x$.
2. $\sum_x P(x) = 1$. That is, the sum of the probabilities for all possible $x$ values is 1.

Example 3: $P(x) = \frac{x}{10}$ for $x = 1, 2, 3, 4$.

Valid Probability Distribution?
Property 1?
Property 2?
Expected Value of a Discrete Random Variable

The expected value of a r.v. is its mean (i.e., the mean of its probability distribution).

For a discrete r.v. $X$, the expected value of $X$, denoted $\mu$ or $E(X)$, is:

$$\mu = E(X) = \sum x P(x)$$

where $\sum$ represents a summation over all values of $x$.

Recall Example 3:

$$\mu =$$

Here, the expected value of $X$ is

Example 4: Suppose a raffle ticket costs $1. Two tickets will win prizes: First prize = $500 and second prize = $300. Suppose 1500 tickets are sold. What is the expected profit for a ticket buyer?

$$x (\text{profit}) P(x)$$

$$E(X) =$$

$E(X) = -0.47$ dollars, so on average, a ticket buyer will lose 47 cents.
The expected value does **not** have to be a possible value of the r.v. --- it’s an *average* value.

**Variance of a Discrete Random Variable**

The variance $\sigma^2$ is the expected value of the squared deviations from the mean $\mu$; that is, $\sigma^2 = \text{E}[(X - \mu)^2]$.

$$\sigma^2 = \sum (x - \mu)^2 P(x)$$

Shortcut formula:

$$\sigma^2 = \left[ \sum x^2 P(x) \right] - \mu^2$$

where $\sum$ represents a summation over all values of $x$.

**Example 3:** Recall $\mu = 3$ for this r.v.

$\sum x^2 P(x) =$

Thus $\sigma^2 =$

Note that the standard deviation $\sigma$ of the r.v. is the square root of $\sigma^2$.

For Example 3, $\sigma =$
The Binomial Random Variable

Many experiments have responses with 2 possibilities (Yes/No, Pass/Fail).

Certain experiments called binomial experiments yield a type of r.v. called a binomial random variable.

Characteristics of a binomial experiment:
  1. The experiment consists of a number (denoted $n$) of identical trials.
  2. There are only two possible outcomes for each trial – denoted “Success” (S) or “Failure” (F).
  3. The probability of success (denoted $p$) is the same for each trial.
     (Probability of failure $= q = 1 – p$.)
  4. The trials are independent.

Then the binomial r.v. (denoted $X$) is the number of successes in the $n$ trials.

Example 1: A fair coin is flipped 5 times. Define “success” as “head”. $X =$ total number of heads.
Then $X$ is

Example 2: A student randomly guesses answers on a multiple choice test with 3 questions, each with 4 possible answers. $X =$ number of correct answers.
Then $X$ is
What is the probability distribution for $X$ in this case?

<table>
<thead>
<tr>
<th>Outcome</th>
<th>$X$</th>
<th>$P$(outcome)</th>
</tr>
</thead>
</table>

**Probability Distribution of $X$**

$x$  \hspace{1cm} $P(x)$
General Formula: (Binomial Probability Distribution)

\( n = \text{number of trials}, \ p = \text{probability of success.} \)

The probability there will be exactly \( x \) successes is:

\[
P(x) = \binom{n}{x} p^x q^{n-x} \quad (x = 0, 1, 2, \ldots, n)
\]

where

\[
\binom{n}{x} = \text{“} n \ \text{choose} \ x \text{”}
\]

\[
= \frac{n!}{x! (n-x)!}
\]

Here, \( 0! = 1, \ 1! = 1, \ 2! = 2 \cdot 1 = 2, \ 3! = 3 \cdot 2 \cdot 1 = 6, \text{ etc.} \)

Example: Suppose probability of “red” in a roulette wheel spin is \( \frac{18}{38} \). In 5 spins of the wheel, what is the probability of exactly 4 red outcomes?
• The mean (expected value) of a binomial r.v. is \( \mu = np \).
• The variance of a binomial r.v. is \( \sigma^2 = npq \).
• The standard deviation of a binomial r.v. is \( \sigma = \sqrt{npq} \).

Example: What is the mean number of red outcomes that we would expect in 5 spins of a roulette wheel?

\[ \mu = np = \]

What is the standard deviation of this binomial r.v.?

Using Binomial Tables
Since hand calculations of binomial probabilities are tedious, Table II gives “cumulative probabilities” for certain values of \( n \) and \( p \).

Example:
Suppose \( X \) is a binomial r.v. with \( n = 10, \ p = 0.40 \).
Table II (page 785) gives:

Probability of 5 or fewer successes: \( P(X \leq 5) = \)

Probability of 8 or fewer successes: \( P(X \leq 8) = \)
What about …

… the probability of exactly 5 successes?

… the probability of more than 5 successes?

… the probability of 5 or more successes?

… the probability of 6, 7, or 8 successes?

Why doesn’t the table give P(X ≤ 10)?
Poisson Random Variables

The Poisson distribution is a common distribution used to model “count” data:
- Number of telephone calls received per hour
- Number of claims received per day by an insurance company
- Number of accidents per month at an intersection

The mean number of events for a Poisson distribution is denoted $\lambda$.

Which values can a Poisson r.v. take?

Probability distribution for $X$
(if $X$ is Poisson with mean $\lambda$)

$$P(x) = \frac{\lambda^x e^{-\lambda}}{x!} \quad (\text{for } x = 0, 1, 2, \ldots)$$

Mean of Poisson probability distribution: $\lambda$

Variance of Poisson probability distribution: $\lambda$
Example: A call center averages 10 calls per hour. Assume $X$ (the number of calls in an hour) follows a Poisson distribution. What is the probability that the call center receives exactly 3 calls in the next hour?

What is the probability the call center will receive 2 or more calls in the next hour?
Calculating Poisson probabilities by hand can be tedious. Table III gives cumulative probabilities for a Poisson r.v., \( P(X \leq k) \) for various values of \( k \) and \( \lambda \).

Example 1: \( X \) is Poisson with \( \lambda = 1 \). Then

\[ P(X \leq 1) = \]
\[ P(X \geq 3) = \]

\[ P(X = 2) = \]

Example 2: \( X \) is Poisson with \( \lambda = 6 \). Then

… probability that \( X \) is 5 or more?

… probability that \( X \) is 7, 8, or 9?
Linear Transformations, Sums, and Differences of Random Variables

In general, the expected value is a “linear operator”. This means:

If:

Then:

In particular:

Also:

Proof:

Hence:

Example: Suppose $X =$ July daily temperature (in degrees Fahrenheit) has mean 92 and standard deviation 2. If $Y =$ July daily temperature (in degrees Celsius), then find the mean and std. deviation of $Y$. 
For two r.v.’s $X$ and $Y$,

If $X$ and $Y$ are independent, then:

**Example 1:** A business undertakes a venture in Atlanta and a venture in Chicago. Assume $X$ (revenue in $ from Atlanta venture) and $Y$ (revenue in $ from Chicago venture) are independent. $X$ has expected value 50,000 and variance 1,000,000. $Y$ has expected value 40,000 and variance 1,000,000.

Find expected total revenue:

Find standard deviation of total revenue:

**Example 2:** Suppose the Atlanta venture has expected cost $35,000 and cost variance = 500,000.

Find expected profit:

Find standard deviation of profit: