Issues with Bayes Factors

- **Note**: When an **improper prior** is used for θ, the Bayes Factor is not well-defined.
- Note $B(\mathbf{x}) = \dfrac{\text{Posterior odds for } M_1}{\text{Prior odds for } M_1}$, and the "prior odds" is meaningless for an improper prior.
- Gill's Sec. 7.3.2 suggests several methods (Local Bayes factors, Intrinsic Bayes Factors, Partial Bayes Factors, Fractional Bayes Factors), none of them ideal, to define types of Bayes Factors with improper priors.
- One criticism of Bayes Factors is the (implicit) assumption that one of the competing models (M_1 or M_2) is correct.
- Another criticism is that the Bayes Factor depends heavily on the choice of prior.

The Bayesian Information Criterion

- The Bayesian Information Criterion (BIC) can be used (as a substitute for the Bayes factor) to compare two (or more) models.
- Conveniently, the BIC does **not** require specifying priors.
- For parameters $\boldsymbol{\theta}$ and data $\mathbf{x}$:

$$BIC = -2 \ln L(\hat{\boldsymbol{\theta}}|\mathbf{x}) + p \ln(n)$$

  where p is the number of free parameters in the model, and $L(\hat{\boldsymbol{\theta}}|\mathbf{x})$ is the **maximized likelihood**, given observed data $\mathbf{x}$.
- Good models have relatively small BIC values:
  - A small value of $-2 \ln L(\hat{\boldsymbol{\theta}}|\mathbf{x})$ indicates good fit to the data;
  - a small value of the "overfitting penalty" term $p \ln(n)$ indicates a simple, parsimonious model.

The Bayesian Information Criterion

▶ To compare two models M_1 and M_2, we could calculate

$$S = -\frac{1}{2}[BIC_{M_1} - BIC_{M_2}]$$
$$= \ln L(\hat{\boldsymbol{\theta}}_1|\mathbf{x}) - \ln L(\hat{\boldsymbol{\theta}}_2|\mathbf{x}) - \frac{1}{2}(p_1 - p_2)\ln(n)$$

▶ A small value of S would favor M_2 here and a large S would favor M_1.

▶ As $n \to \infty$,

$$\frac{S - \ln(B(\mathbf{x}))}{\ln(B(\mathbf{x}))} \to 0$$

and for large n,

$$BIC_{M_1} - BIC_{M_2} = -2S \approx -2\ln(B(\mathbf{x})).$$

- Note that differences in *BIC*'s can be used to compare several nonnested models.
- They should be trusted as a substitute for Bayes Factors only when (1) no reliable prior information is available and (2) when the sample size is **quite large**.
- See R examples: (1) Calcium data example and (2) Regression example on Oxygen Uptake data set.