# Frequentist Coverage for Bayesian Intervals

▶ Hartigan (1966) showed that for standard posterior intervals, an interval with  $100(1-\alpha)\%$  Bayesian coverage will have

$$P[L(\mathbf{X}) < \theta < U(\mathbf{X})|\theta] = (1 - \alpha) + \epsilon_n,$$

where  $|\epsilon_n| < a/n$  for some constant a.

- $\Rightarrow$  Frequentist coverage  $\rightarrow 1 \alpha$  as  $n \rightarrow \infty$ .
- Note that many classical CI methods only achieve  $100(1-\alpha)\%$  frequentist coverage asymptotically, as well.

# Bayesian Credible Intervals

- ► A **credible interval** (or in general, a **credible set**) is the Bayesian analogue of a confidence interval.
- ▶ A  $100(1-\alpha)$ % credible set C is a subset of  $\Theta$  such that

$$\int_{\mathcal{C}}\pi(oldsymbol{ heta}|\mathbf{X})\,doldsymbol{ heta}=1-lpha.$$

▶ If the parameter space  $\Theta$  is discrete, a sum replaces the integral.

#### Quantile-Based Intervals

▶ If  $\theta_L^*$  is the  $\alpha/2$  posterior quantile for  $\theta$ , and  $\theta_U^*$  is the  $1 - \alpha/2$  posterior quantile for  $\theta$ , then  $(\theta_L^*, \theta_U^*)$  is a  $100(1 - \alpha)\%$  credible interval for  $\theta$ .

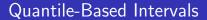
Note: 
$$P[\theta < \theta_L^* | \mathbf{X}] = \alpha/2$$
 and  $P[\theta > \theta_U^* | \mathbf{X}] = \alpha/2$ .  

$$\Rightarrow P\{\theta \in (\theta_L^*, \theta_U^*) | \mathbf{X}\}$$

$$= 1 - P\{\theta \notin (\theta_L^*, \theta_U^*) | \mathbf{X}\}$$

$$= 1 - \left(P[\theta < \theta_L^* | \mathbf{X}] + P[\theta > \theta_U^* | \mathbf{X}]\right)$$

$$= 1 - \alpha.$$



Picture:

- ▶ Suppose  $X_1, ..., X_n$  are the durations of cabinets for a sample of cabinets from Western European countries.
- ▶ We assume the  $X_i$ 's follow an exponential distribution.

$$p(X_i|\theta) = \theta e^{-\theta X_i}, X_i > 0$$
  
$$\Rightarrow L(\theta|\mathbf{X}) = \theta^n e^{-\theta \sum_{i=1}^n x_i}$$

Suppose our prior distribution for  $\theta$  is

$$p(\theta) \propto 1/\theta, \ \theta > 0.$$

 $\Rightarrow$  Larger values of  $\theta$  are less likely **a priori**.

Then

$$\pi( heta|\mathbf{X}) \propto p( heta)L( heta|\mathbf{X}) \ \propto \left(rac{1}{ heta}
ight) heta^n e^{- heta \sum x_i} \ = heta^{n-1} e^{- heta \sum x_i}$$

- ▶ This is the **kernel** of a **gamma** distribution with "shape" parameter n and "rate" parameter  $\sum_{i=1}^{n} x_i$ .
- So including the normalizing constant,

$$\pi(\theta|\mathbf{X}) = \frac{(\sum x_i)^n}{\Gamma(n)} \theta^{n-1} e^{-\theta \sum x_i}, \ \theta > 0.$$

- Now, given the observed data  $x_1, \ldots, x_n$ , we can calculate any quantiles of this gamma distribution.
- ▶ The 0.05 and 0.95 quantiles will give us a 90% credible interval for  $\theta$ .
- See R example with real data on course web page.

- ▶ Suppose we feel  $p(\theta) = 1/\theta$  is too subjective and favors small values of  $\theta$  too much.
- Instead, let's consider the noninformative prior

$$p(\theta) = 1, \ \theta > 0$$

(favors all values of  $\theta$  equally).

► Then our posterior is

$$egin{aligned} \pi( heta|\mathbf{X}) &\propto p( heta) L( heta|\mathbf{X}) \ &= (1) heta^n e^{- heta \sum x_i} \ &= heta^{(n+1)-1} e^{- heta \sum x_i} \end{aligned}$$

- $\Rightarrow$  This posterior is a gamma with parameters (n+1) and  $\sum x_i$ .
- We can similarly find the equal-tail credible interval.

- ▶ Consider 10 flips of a coin having  $P\{\text{Heads}\} = \theta$ .
- ▶ Suppose we observe 2 "heads".
- We model the count of heads as binomial:

$$p(X|\theta) = {10 \choose X} \theta^X (1-\theta)^{10-X}, \quad x = 0, 1, \dots, 10.$$

Let's use a uniform prior for  $\theta$ :

$$p(\theta) = 1, 0 \le \theta \le 1.$$

► Then the posterior is:

$$\pi(\theta|x) \propto p(\theta)L(\theta|x)$$

$$= (1) \binom{10}{x} \theta^{x} (1-\theta)^{10-x}$$

$$\propto \theta^{x} (1-\theta)^{10-x}, \quad 0 \le \theta \le 1.$$

- ► This is a **beta** distribution for  $\theta$  with parameters x + 1 and 10 x + 1.
- ▶ Since x = 2 here,  $\pi(\theta|x = 2)$  is beta(3,9).
- ▶ The 0.025 and 0.975 quantiles of a beta(3,9) are (.0602, .5178), which is a 95% credible interval for  $\theta$ .