Hartigan (1966) showed that for standard posterior intervals, an interval with $100(1 - \alpha)\%$ Bayesian coverage will have

$$P[L(X) < \theta < U(X) | \theta] = (1 - \alpha) + \epsilon_n,$$

where $|\epsilon_n| < a/n$ for some constant $a$.

$\Rightarrow$ Frequentist coverage $\rightarrow 1 - \alpha$ as $n \rightarrow \infty$.

Note that many classical CI methods only achieve $100(1 - \alpha)\%$ frequentist coverage asymptotically, as well.
A **credible interval** (or in general, a **credible set**) is the Bayesian analogue of a confidence interval.

A $100(1 - \alpha)\%$ credible set $C$ is a subset of $\Theta$ such that

$$\int_C \pi(\theta|X) \, d\theta = 1 - \alpha.$$

If the parameter space $\Theta$ is discrete, a sum replaces the integral.
Quantile-Based Intervals

If $\theta^*_L$ is the $\alpha/2$ posterior quantile for $\theta$, and $\theta^*_U$ is the $1 - \alpha/2$ posterior quantile for $\theta$, then $(\theta^*_L, \theta^*_U)$ is a $100(1 - \alpha)%$ credible interval for $\theta$.

Note: $P[\theta < \theta^*_L|X] = \alpha/2$ and $P[\theta > \theta^*_U|X] = \alpha/2$.

$$\Rightarrow P\{\theta \in (\theta^*_L, \theta^*_U)|X\}$$
$$= 1 - P\{\theta \notin (\theta^*_L, \theta^*_U)|X\}$$
$$= 1 - \left( P[\theta < \theta^*_L|X] + P[\theta > \theta^*_U|X] \right)$$
$$= 1 - \alpha.$$
Quantile-Based Intervals

Picture:
Suppose $X_1, \ldots, X_n$ are the durations of cabinets for a sample of cabinets from Western European countries.

We assume the $X_i$’s follow an exponential distribution.

$$p(X_i|\theta) = \theta e^{-\theta X_i}, \quad X_i > 0$$

$$\Rightarrow L(\theta|X) = \theta^n e^{-\theta \sum_{i=1}^{n} x_i}$$

Suppose our prior distribution for $\theta$ is

$$p(\theta) \propto 1/\theta, \quad \theta > 0.$$  

$$\Rightarrow$$ Larger values of $\theta$ are less likely a priori.
Then

\[ \pi(\theta|X) \propto p(\theta)L(\theta|X) \]
\[ \propto \left(\frac{1}{\theta}\right)^\theta e^{-\theta \sum x_i} \]
\[ = \theta^{n-1} e^{-\theta \sum x_i} \]

- This is the **kernel** of a **gamma** distribution with “shape” parameter \( n \) and “rate” parameter \( \sum_{i=1}^{n} x_i \).
- So including the normalizing constant,

\[ \pi(\theta|X) = \frac{(\sum x_i)^n}{\Gamma(n)} \theta^{n-1} e^{-\theta \sum x_i}, \quad \theta > 0. \]
Now, given the observed data $x_1, \ldots, x_n$, we can calculate any quantiles of this gamma distribution.

The 0.05 and 0.95 quantiles will give us a 90% credible interval for $\theta$.

See R example with real data on course web page.
Example: Quantile-Based Interval

- Suppose we feel $p(\theta) = 1/\theta$ is too subjective and favors small values of $\theta$ too much.
- Instead, let’s consider the noninformative prior

  \[
p(\theta) = 1, \quad \theta > 0
  \]

  (favors all values of $\theta$ equally).
- Then our posterior is

  \[
  \pi(\theta|X) \propto p(\theta)L(\theta|X)
  \]

  \[
  = (1)\theta^n e^{-\theta \sum x_i}
  \]

  \[
  = \theta^{(n+1)-1} e^{-\theta \sum x_i}
  \]

  $\Rightarrow$ This posterior is a gamma with parameters $(n + 1)$ and $\sum x_i$.
- We can similarly find the equal-tail credible interval.
Example 2: Quantile-Based Interval

- Consider 10 flips of a coin having $P\{\text{Heads}\} = \theta$.
- Suppose we observe 2 “heads”.
- We model the count of heads as binomial:

$$p(X|\theta) = \binom{10}{X} \theta^X (1 - \theta)^{10-X}, \quad x = 0, 1, \ldots, 10.$$  

- Let’s use a uniform prior for $\theta$:

$$p(\theta) = 1, \quad 0 \leq \theta \leq 1.$$
Example 2: Quantile-Based Interval

Then the posterior is:

\[ \pi(\theta|x) \propto p(\theta)L(\theta|x) \]

\[ = (1)\binom{10}{x} \theta^x (1 - \theta)^{10-x} \]

\[ \propto \theta^x (1 - \theta)^{10-x}, \quad 0 \leq \theta \leq 1. \]

This is a **beta** distribution for \( \theta \) with parameters \( x + 1 \) and \( 10 - x + 1 \).

Since \( x = 2 \) here, \( \pi(\theta|x = 2) \) is beta(3,9).

The 0.025 and 0.975 quantiles of a beta(3,9) are (.0602, .5178), which is a 95% credible interval for \( \theta \).