"Treatments" $\rightarrow$ factor levels (in one-way ANOVA) or factor level combinations (in multi-factor ANOVA)

**Design of Experiments**

- Factorial experiments require a lot of resources

- Sometimes real-world practical considerations require us to design experiments in specialized ways.

- The **design** of an experiment is the specification of how treatments are assigned to experimental units.

**Goal:** Gain maximum amount of reliable information using minimum amount of resources.

- Reliability of information is measured by the **standard error** of an estimate.

- How to decrease standard errors and thereby increase reliability?
  - Increase sample size *(costly, sometimes impractical)*
  - Decrease population variance $\sigma^2$

- Recall the One-Way ANOVA:

- Experiments we studied used the Completely Randomized Design (CRD).
**Example 3:** An industrial experiment is conducted over several days (with a different lab technician each day).
- Possible block design:

Then the technicians (or the days) could be blocks.

**Example 4:** (Table 10.2 data)

- $Y =$ wheat crop yield
- experimental units = plots of wheat
- treatments = 3 different varieties of wheat $(A, B, C)$
- blocks = regions of field

Possible arrangement:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>C</td>
<td>A</td>
<td>C</td>
<td>B</td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>B</td>
<td>B</td>
<td>A</td>
<td>C</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>A</td>
<td>C</td>
<td>B</td>
<td>A</td>
<td></td>
</tr>
</tbody>
</table>
• The data are given in Table 10.2.

• Note: Variety A has the greatest mean yield, but there is a sizable variation among blocks.

• If we had used a CRD, this variation would all be experimental error variance (inflates MSW).

• Analysis as CRD (ignoring blocks):

\[ F^* = \frac{49.217}{13.608} = 3.62 \quad (p-value = .059) \]

So \( \alpha = .05 \), we do not conclude the mean yield significantly differs across the 3 varieties.

• But ... within each block, Variety A clearly has the greatest yield (RBD will account for this).
Formal Linear Model for RBD

\[ Y_{ij} = \mu + \tau_i + \beta_j + \varepsilon_{ij} \quad i = 1, \ldots, t \]
\[ j = 1, \ldots, b \]

- This assumes **one observation per treatment-block combination**.

\( Y_{ij} = \) response value for treatment \( i \) in block \( j \)
\( \mu = \) an overall mean response
\( \tau_i = \) effect of treatment \( i \)
\( \beta_j = \) effect of block \( j \)
\( \varepsilon_{ij} = \) random error term

- Looks similar to two-factor factorial model with one observation per cell. (assume no treatment \( \times \) block interaction)

**Key difference:** With RBD, we are not equally interested in both factors.
- The treatment factor is of primary importance; the blocking factor is included merely to reduce experimental error variance.

- With RBD, the block effects are often considered random (not fixed) effects.

- This is true if the blocks used are a random sample from a large population of possible blocks.
- If treatment effects are fixed and block effects are random, the RBD model is called a **mixed model**.

- In this case, the treatment-block interaction is also random.

- This interaction measures the variation among treatment effects across the various blocks.

- The mean square for interaction is used here as an estimate of the **experimental error variance** $\sigma^2$.

### Expected Mean Squares in RBD

<table>
<thead>
<tr>
<th>Source</th>
<th>df</th>
<th>E(MS)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Treatments</td>
<td>$t-1$</td>
<td>$\sigma^2 + \frac{b}{t-1} \sum \tau_i^2$</td>
</tr>
<tr>
<td>Blocks</td>
<td>$b-1$</td>
<td></td>
</tr>
<tr>
<td>Exper. Error</td>
<td>$(t-1)(b-1)$</td>
<td>$\sigma^2 + t \sigma_B^2$</td>
</tr>
</tbody>
</table>

\( (Trt \times Block Interaction) \)

\[ \sigma^2 = \text{experimental error variance} \]

\[ \sigma_B^2 = \text{variance among block effects} \]
• Testing for an effect on the mean response among treatments:

\[ H_0: \tau_1 = \tau_2 = \cdots = \tau_t = 0 \iff \sum_i \tau_i^2 = 0 \]

• The correct test statistic is apparent based on \( E(\text{MS}) \):

\[ F^* = \frac{\text{MS(Trts)}}{\text{MSE}} \]

Reject \( H_0 \) if:

\[ F^* > F_\alpha \left[ t-1, (t-1)(b-1) \right] \]

• Testing for significant variation across blocks:

\[ H_0: \sigma_B^2 = 0 \quad H_a: \sigma_B^2 > 0 \]

• The correct test statistic is again apparent:

\[ F^* = \frac{\text{MS(Blocks)}}{\text{MSE}} \]

Reject \( H_0 \) if:

\[ F^* > F_\alpha \left[ (b-1), (t-1)(b-1) \right] \]

Example: (Wheat data – Table 10.2)

• The ANOVA table formulas are the same as for the two-way ANOVA.

• We use software for the ANOVA table computations.
RBD analysis (Wheat data):

\[ F^* = \frac{MS(Trts)}{MSE} = \frac{49.217}{1.8} = 27.34 \quad (P\text{-value} = .0003) \]

- We conclude that the mean yields are significantly different for the different varieties of wheat. At \( \alpha = 0.05 \), we reject \( H_0: \tau_1 = \tau_2 = \tau_3 = 0 \).

Note (for testing about blocks):

\[ F^* = \frac{MS(Blocks)}{MSE} = \frac{37.225}{1.8} = 20.68 \quad (P\text{-value} = .0003) \]

- We would also reject \( H_0: \sigma^2 = 0 \) and conclude there is significant variation among block effects.

- We can again make pre-planned comparisons using contrasts.

**Example:** Is Variety A **superior** to the other two varieties in terms of mean yield?

\[ L = \mu_A - \frac{1}{2} \mu_B - \frac{1}{2} \mu_C \]

\[ H_0: \quad \mu_A - \frac{1}{2} \mu_B - \frac{1}{2} \mu_C = 0 \]

\[ H_a: \quad \mu_A - \frac{1}{2} \mu_B - \frac{1}{2} \mu_C > 0 \]

Result: \( t^* = 7.28 \) (evidence in favor of \( H_a \))

\( t^* = 7.28 > 1.86 = t_{.05, 8 \text{ d.f.}} \Rightarrow \text{reject } H_0, \text{ conclude } H_a. \)

SAS gives two-sided p-value of \( < .0001 \).

\( \Rightarrow \text{One-sided p-value here is } < \frac{.0001}{2} \Rightarrow < .00005 \)

\( \Rightarrow \text{Reject } H_0, \text{ conclude Variety A is superior in terms of mean yield.} \)
• The estimate of $\sigma^2$ was MSW. This measured the variation among responses for units that were treated alike (measured variation within groups).

• We call this estimating the experimental error variation.

• What if we divide the units into subgroups (called blocks) such that units within each subgroup were similar in some way?

• We would expect the variation in response values among units treated alike within each block to be relatively small.

**Randomized Block Design (RBD)**

• RBD: A design in which experimental units are divided into subgroups called blocks and treatments are randomly assigned to units within each block.

• Blocks should be chosen so that units within a block are similar in some way.

• Reasons for the variation in our data values:

  CRD (Chap. 6)
  - Variation due to treatments (levels)
  - Experimental error variation (leftover variation)

  RBD (Chap. 10)
  - Variation due to treatments
  - Variation due to blocks
  - Experimental error variation (leftover) now reduced
• Benefits of a reduction in experimental error:
  • decreases MSW (denominator of $F^*$ ratios used in $F$-tests) $\rightarrow$ more power to reject null hypotheses
  • decreases standard errors of means $\rightarrow$ shorter CIs for mean responses

Example 1: Suppose we investigate whether the average math-test scores of students from 8 different majors differ across majors.
  • But … students will be taught by different instructors.
  • We’re not as interested in the instructor effect, but we know it adds another layer of variability.

Solution: Make "instructors" the blocks
  units = students
  (response) $Y =$ test score
  treatments = 8 majors
  blocks = the instructors

Example 2: Lab animals of a certain species are given different diets to determine the effect of diet on weight gain.
  • Possible block design:
    units = animals
    $Y =$ weight gain
    treatments = diets
    blocks = litters the animals were born into