3) Suppose $\{e_t\}$ is a normal white noise process with mean zero and variance σ_e^2. Let $\{Y_t\}$ be a process defined as: $Y_t = e_t + \theta e_{t-1}$.

a) Find the autocovariance function and autocorrelation function of Y_t for any general θ. Also, find the autocovariance function and autocorrelation function of Y_t if $\theta = 2$. Show all your steps clearly.

b) Is the time series $\{Y_t\}$ stationary? Explain your answer.

a) We calculate $\operatorname{cov}(Y_t, Y_{t-k})$ for 3 cases: $k=0$, $k=1$, and $k>1$.

$$\operatorname{cov}(Y_t, Y_t) = \operatorname{var}(Y_t) = \operatorname{var}(e_t + \theta e_{t-1}) = \operatorname{var}(e_t) + \theta^2 \operatorname{var}(e_{t-1})$$
$$= \sigma_e^2 + \theta^2 \sigma_e^2$$

$$\operatorname{cov}(Y_t, Y_{t-1}) = \operatorname{cov}(e_t + \theta e_{t-1}, e_{t-1} + \theta e_{t-2}) = \operatorname{cov}(e_t, e_{t-1})$$
$$+ \theta \operatorname{cov}(e_t, e_{t-2}) + \theta \operatorname{cov}(e_{t-1}, e_{t-1}) + \theta^2 \operatorname{cov}(e_{t-1}, e_{t-2})$$
$$= 0 + 0 + \theta \sigma_e^2 + 0 = \theta \sigma_e^2$$

and $\operatorname{cov}(Y_t, Y_{t-k})$ for any $k \geq 2$ is easily shown to be 0 since there are no overlapping terms.

Writing the lag k as $t-s$, we can summarize the ~~auto~~ auto covariance function $\gamma_{t,s}$ as:

$$\gamma_{t,s} = \operatorname{cov}(Y_t, Y_s) = \begin{cases} \sigma_e^2 + \theta^2 \sigma_e^2 & \text{if } |t-s| = 0 \\ \theta \sigma_e^2 & \text{if } |t-s| = 1 \\ 0 & \text{if } |t-s| > 1 \end{cases}$$

Clearly $\mathrm{corr}(Y_t, Y_t) = 1$ and $\mathrm{corr}(Y_t, Y_{t-k}) = 0$ for $k > 1$. When $k = 1$,

$$\mathrm{corr}(Y_t, Y_{t-1}) = \frac{\mathrm{cov}(Y_t, Y_{t-1})}{\sqrt{\mathrm{var}(Y_t)\,\mathrm{var}(Y_{t-1})}} = \frac{\theta\sigma_e^2}{\sqrt{(\sigma_e^2 + \theta^2\sigma_e^2)(\sigma_e^2 + \theta^2\sigma_e^2)}}$$

$$= \frac{\theta\sigma_e^2}{\sigma_e^2 + \theta^2\sigma_e^2} = \frac{\theta}{1+\theta^2}$$

So the autocorrelation function $\rho_{t,s}$ is

$$\rho_{t,s} = \mathrm{corr}(Y_t, Y_s) = \begin{cases} 1 & \text{if } |t-s| = 0 \\ \frac{\theta}{1+\theta^2} & \text{if } |t-s| = 1 \\ 0 & \text{if } |t-s| > 1 \end{cases}$$

If $\theta = 2$:

$$\gamma_{t,s} = \begin{cases} 5\sigma_e^2 & \text{if } |t-s| = 0 \\ 2\sigma_e^2 & \text{if } |t-s| = 1 \\ 0 & \text{if } |t-s| > 1 \end{cases} \qquad \rho_{t,s} = \begin{cases} 1 & \text{if } |t-s| = 0 \\ 2/5 & \text{if } |t-s| = 1 \\ 0 & \text{if } |t-s| > 1 \end{cases}$$

b) Yes, $\{Y_t\}$ is stationary: The mean function $E(Y_t) = 0$ and so does not depend on t.
The variance function $\mathrm{var}[Y_t] = \theta\sigma_e^2$, which does not depend on t. The autocovariance $\mathrm{cov}(Y_t, Y_s)$ depends only on the lag between t and s.
~~And the pr~~ Thus the process is weakly stationary, and since the process follows a normal distribution, $\{Y_t\}$ is stationary.

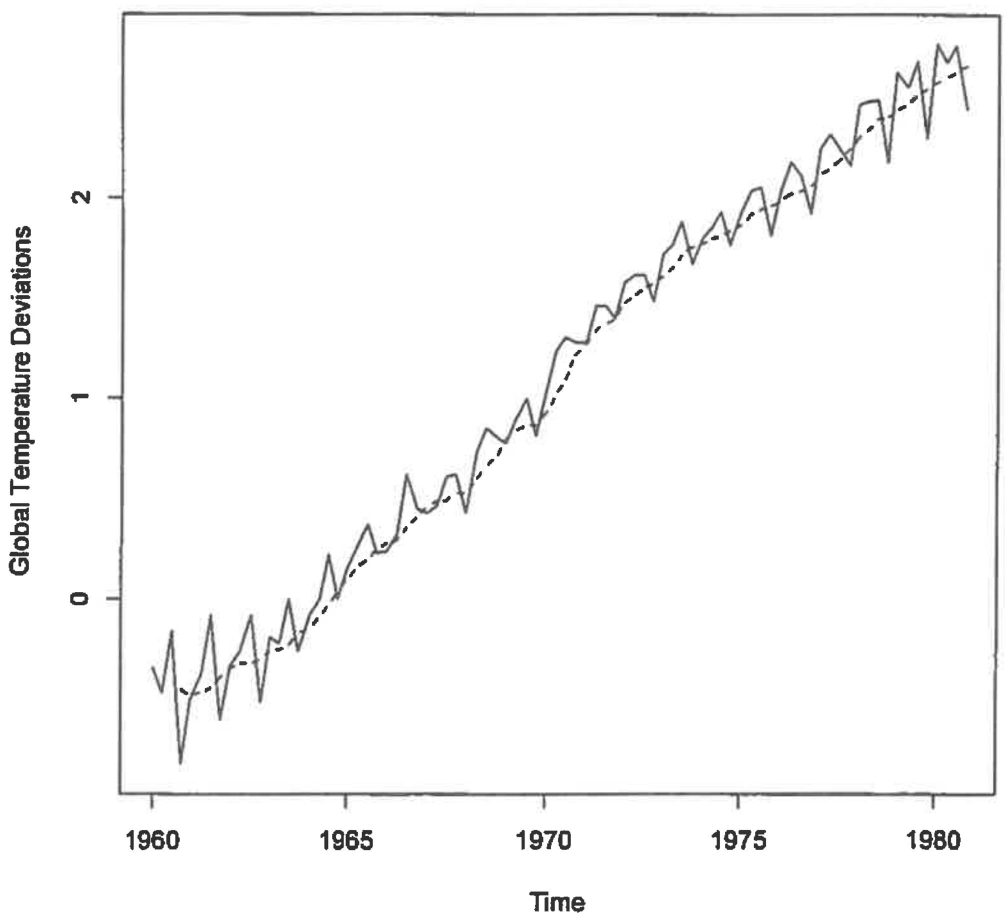
4) Apply a moving average filter to Y_t, where Y_t is the natural logarithm of the Johnson and Johnson earnings data (the original data are given in the `jj` object in the `astsa` package). Specifically, let $V_t = (Y_t + Y_{t-1} + Y_{t-2} + Y_{t-3}) / 4$. The R code

```
v = filter(y, rep(1/4, 4), sides = 1)
```

may be helpful in implementing this. Type `help(filter)` in R for more details about this R function. Plot Y_t as a line and overlay (superimpose) V_t as a dashed line, and provide this plot. Discuss whether the moving average filter captures the overall trend in the time series.

R code:

```
library(astsa)
y = log(jj)
v = filter(y, rep(1/4,4), sides=1)
plot(y, type="l", ylab="Global Temperature Deviations")
lines(v, lty=2)
```



We see that the moving average filter captures the basic increasing trend of the plot quite well. The mean of the logged earnings increases almost linearly, but less steeply in the early 1960s and more steeply in the years around 1970.

5) [Required for graduate students, extra credit for undergraduate students] Suppose $\{e_t\}$ is a normal white noise process with mean zero and variance σ_e^2. Let $\{Y_t\}$ be a process defined as:
$Y_t = e_t e_{t-1}$. Showing all your steps, find the mean function and the autocovariance function of Y_t. Is the time series $\{Y_t\}$ stationary? Explain your answer.

$$\mu_t = E[Y_t] = E[e_t e_{t-1}] = E[e_t]E[e_{t-1}] \text{ by independence}$$
$$= 0$$

$$\mathrm{cov}(Y_t, Y_t) = \mathrm{var}(Y_t) = E[(Y_t - \mu_t)^2] = E[Y_t^2]$$
$$= E[e_t^2 e_{t-1}^2] = E[e_t^2]\, E[e_{t-1}^2]$$
$$= \{\mathrm{var}(e_t) + [E(e_t)]^2\}\{\mathrm{var}(e_{t-1}) + [E(e_{t-1})]^2\}$$
$$= (\sigma_e^2 + 0)(\sigma_e^2 + 0) = \sigma_e^4$$

$$\mathrm{cov}(Y_t, Y_{t-1}) = \mathrm{cov}[e_t e_{t-1}, e_{t-1} e_{t-2}]$$
$$= E[e_t e_{t-1}^2 e_{t-2}] - E[e_t e_{t-1}]\, E[e_{t-1} e_{t-2}]$$
$$= E[e_t]\, E[e_{t-1}^2]\, E[e_{t-2}] - E[e_t]E[e_{t-1}]\, E[e_{t-1}]\, E[e_{t-2}]$$
$$= (0)[E(e_{t-1}^2)](0) - 0 = 0$$

For $k > 1$,

$$\mathrm{cov}[Y_t, Y_{t-k}] = \mathrm{cov}[e_t e_{t-1}, e_{t-k} e_{t-k-1}]$$
$$= E[e_t e_{t-1} e_{t-k} e_{t-k-1}] - E[e_t e_{t-1}]\, E[e_{t-k} e_{t-k-1}]$$

which is 0 since the e_i's are all independent and all have expected value 0.

The time series is stationary since the mean function does not depend on time and for any k, $\mathrm{cov}(Y_t, Y_{t-k})$ does not depend on t, and the process is normally distributed.

$$\gamma_{t,s} = \begin{cases} \sigma_e^4 & \text{if } |t-s| = 0 \\ 0 & \text{if } |t-s| = 1 \\ 0 & \text{if } |t-s| > 1 \end{cases}$$