4.6 Mean Time Spent in Transient States

- Consider a finite state Markov chain.
- Number the states so that
  is the set of transient states.
- Let

be the matrix with transition probabilities from transient states into transient states.

- This is not a full transition probability matrix (its rows do not all sum to 1).
- For transient states i and j, let $S_{ij}$ be the expected number of time periods that the chain is in j, given that it starts in i.
Define \[ S_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \]

Then

Note we need to sum over only the transient states \( k = 1, \ldots, t \) since the chain cannot go from a recurrent state to a transient state. Why not?

Let the matrix \( S \) contain all \( \{ S_{ij} \} \), \( i, j = 1, \ldots, t \):

\[
S = \]

- Note that in matrix notation,

- So given $P_T$, the $S_{ij}$ values are easily calculated.

**Roulette Example again**: What is the expected amount of time the gambler has $80$?

- Note letting $1$ unit $= 10$,

$$P_T = \ldots$$

- It is easy to calculate $(I-P_T)^{-1}$ in R.
- We want
- What is the expected amount of time the gambler has $20$?

- Note for transient states $i$ and $j$, $f_{ij}$ is the probability the chain ever enters $j$ given that it starts in $i$.

- Conditioning on whether the chain enters $j$, we have:
These probabilities can be found from $S$, which can be found from $P_T$.

Previous Example: What is the probability the roulette gambler ever has exactly $90$?

What is the expected amount of time the gambler plays the game?