- We can use this result to verify the answer in Example 1(a).

Theorem: If $X_1, \ldots, X_n$ are independent exponential r.v.'s with respective rates $\lambda_1, \ldots, \lambda_n$, then:

(1) $\min_i X_i$ is exponential with rate $\sum_{i=1}^{n} \lambda_i$

and (2) $\min_i X_i$ and the rank order of $X_1, \ldots, X_n$ are independent.

Proof of (1):

Proof of (2):

Corollary: If $X_1, \dots, X_n$ are independent exponential r.v.'s with respective rates $\lambda_1, \dots, \lambda_n$, then for any $i \in \{1, \dots, n\}$:

Example 1(b): You are first in line at the post office. There are two clerks, each busy with a customer. The clerks' service times are exponential with rates λ_1 and λ_2. What is the expected time you will spend in the post office?

- If (as in Example 1(a)), both clerks have mean 5 minutes service time, then your expected time in the post office is:

Sums of Exponential r.v.'s

- Let $X_1, \ldots, X_n$ be independent exponential r.v.'s with rates $\lambda_1, \ldots, \lambda_n$, where $\lambda_i \neq \lambda_j$ for $i \neq j$.
- The r.v. $\sum_{i=1}^{n} X_i$ is called a <u>hypoexponential</u> r.v.
- Consider the $n=2$ case. The pdf of $X_1 + X_2$ can be found using the convolution formula (2.18) on page 53:

$$f_{X_1+X_2}(t) = \int_0^t f_{X_1}(s) f_{X_2}(t-s)\, ds$$

$$=$$

Extending this to $n=3$, it can be shown:

and for general n:

which can be shown formally via mathematical induction.

Coxian Random Variables

- Let $X_1, \ldots, X_m$ be independent exponential r.v.'s with rates $\lambda_1, \ldots, \lambda_m$ ($\lambda_i \neq \lambda_j$ for $i \neq j$).
- Let r.v. $N \in \{1, 2, \ldots, m\}$ be independent of $X_1, \ldots, X_m$ and let $P_n = P[N = n]$, so that $\sum_{n=1}^{m} P_n = 1$. Then $Y = \sum_{j=1}^{N} X_j$ is a <u>Coxian</u> r.v. Its pdf is found by conditioning:

<u>Example 2(a):</u> Suppose a patient must go through 3 stages of a program to be cured. The times spent in each stage are independent exponential r.v.'s

with means 16 days, 20 days, and 10 days. (So $\lambda_1 =$, $\lambda_2 =$, $\lambda_3 =$).
If she is certain to remain in the program until being cured, find the probability that her total time in the program is less than 30 days.

Example 2(b): Same situation, but now suppose the patient has probability 0.2 of dropping out after stage 1 and probability 0.3 of dropping out after stage 2. What is the probability that her total time in the program is less than 30 days?