5.3 The Poisson Process

Defn. A stochastic process \{N(t), t \geq 0\} is called a counting process if \(N(t)\) represents the total number of events that have occurred by time \(t\).

Example 1: \(N(t)\) = total number of customers entering a store by time \(t\).

Example 2: \(N(t)\) = total number of people in a population who have been born by time \(t\).

Properties of a Counting Process

1. \(N(t) \geq 0\) for all \(t\)
2. \(N(t)\) is integer-valued.
3. If \(s < t\), then \(N(s) \leq N(t)\).
4. For \(s < t\), then \(N(t) - N(s)\) =
Note: “Number of customers in a store at time t” and “Number of people alive at time t” are not counting processes. Why not?

- A counting process has independent increments if the numbers of events occurring in disjoint time intervals are independent.

- For example, if

- Which example (1 or 2) is more likely to have independent increments?

- A counting process has stationary increments if the distribution of the number of events in an interval depends only on the length of the interval, i.e., if for any fixed $t > 0$, 
- Example 1 would have stationary increments only if:

Poisson Process
- An important counting process is the Poisson Process.

Defn. ("Little-oh" notation) A function \( f(\cdot) \) is \( o(h) \) if \( \lim_{h \to 0} \frac{f(h)}{h} = 0 \).

- Simply put, \( f(\cdot) \) is \( o(h) \) if \( f(h) \) goes to 0 faster than \( h \) does.

Examples: \( f(x) = x^2 \) is

\[ f(x) = x \] is
- If $f(\cdot)$ is $o(h)$ and $g(\cdot)$ is $o(h)$, then $c_1 f(\cdot) + c_2 g(\cdot)$ is $o(h)$ for any constants $c_1, c_2$.

**Defn. (Poisson Process):** A counting process $\{N(t)\}$ is a **Poisson process** with rate $\lambda > 0$ if:

(i) $N(0) = 0$
(ii) $\{N(t)\}$ has independent increments
(iii) $P[N(t+h) - N(t) = 1] = \lambda h + o(h)$
(iv) $P[N(t+h) - N(t) \geq 2] = o(h)$

**Lemma (Poisson approximation to binomial):**
- Let $X \sim \text{binom}(n, p)$ and suppose $n \to \infty$ and $p \to 0$ such that $\lambda = np$ is constant.
  Then for $i = 0, 1, 2, \ldots$, 
Proof:

Hence for n large and p small, a binomial probability can be approximated by a Poisson probability.

**Theorem:** If \( \{N(t)\} \) is a Poisson process with rate \( \lambda > 0 \), then for all \( s > 0, t > 0 \),

\[
N(s+t) - N(s) \sim \text{Poisson}(\lambda t).
\]
- This implies the number of events in any interval of length $t$ is Poisson ($\lambda t$).

Proof: We first note: If we fix $s$ and define $N_s(t)$ to be the count of events in the first $t$ time units past $s$, then it is clear that
Interarrival and Waiting Time Distributions

- In a Poisson Process, let $T_1$ be the time of the first event.
- For $n=2,3,...$, let $T_n$ be the time elapsed between the $(n-1)$-st and $n$-th events.
- Then $\{T_n, n=1,2,...\}$ is the sequence of interarrival times.
- Let's derive the distribution of $T_1$: $P[T_1 > t] =$
- Hence $T_1$ is:
- And $P[T_2 > t] =$

So $T_2$ is
- This argument can be repeated for all \( \{T_n\} \), showing:

**Theorem:** \( T_n \) \( n=1,2,\ldots \) are iid exponential r.v.'s, each with rate \( \lambda \).

**Note:** The assumption of stationary and independent increments implies that the process is memoryless, so having exponential interarrival times is unsurprising.

**Defn.** The waiting time until the \( n \)-th event (also called the arrival time of the \( n \)-th event) is defined as:

\[
S_n = \sum_{i=1}^{n} T_i \quad \text{(for } n=1,2,\ldots) \]

- Since the \( T_i \)'s are iid exponential (\( \lambda \)), then \( S_n \sim \) \[\text{_______} \] with pdf:
Note also the cdf of $S_n$ can be derived:

Then differentiation of $F_{S_n}(t)$ yields the __________ pdf.

**Example 1:** Suppose customers enter a store following a Poisson process with rate $\lambda = 25$ per hour.

What is the probability of having exactly 3 customers in the first 15 minutes? How about 3 or fewer in the first 15 minutes?
-What is the expected time until the 40th customer arrives?

-What is the probability that the next customer arrives within three minutes after this 40th customer?