Chapter 6: Continuous-Time Markov Chains

- In Chapter 4, we studied Markov chains \( \{X_n\} \) with a discrete index set \( n = 0, 1, 2, \ldots \).
- We now study continuous-time Markov chains \( \{X(t), t \geq 0\} \) in which the index set is all non-negative real numbers.
- These processes have the Markovian property: Their future behavior depends only on the present, not the past.

6.2 Continuous-Time Markov Chains

**Defn:** A stochastic process \( \{X(t), t \geq 0\} \) whose state space is the set of non-negative integers, is a continuous-time Markov chain if
for all $s, t \geq 0$ and all $i, j, x(u), 0 \leq u \leq s$:

- Hence the conditional distribution of the process at time $t+s$ depends only on the state at time $s$, for all $s, t \geq 0$.

- If $P[X(t+s) = j \mid X(s) = i]$ does not depend on $s$, then the chain has

  transition probabilities.

Note: If the chain enters state $i$, let $T_i$ denote the amount of time it spends in $i$ before transitioning to a new state.

- By the Markovian property, future behavior only depends on where the chain is currently.
Therefore:

- This implies that \( T_i \) is ______ and so it must be distributed as ______.

- Hence we can alternately define a continuous-time Markov chain as a process which, when it enters some state \( i \):
  
  (1)

  and (2)

- Note that which state \( j \) is visited after state \( i \) cannot depend on the amount of time spent in state \( i \) (otherwise, this would violate the Markovian property).
Example: A drive-through restaurant has two windows, one for ordering food and one for paying and picking up the food. Suppose the service times (in minutes) at windows 1 and 2 are independent exponential r.v.'s, with rates 1 and $\frac{1}{3}$, respectively. Suppose customers arrive at the restaurant according to a Poisson process with rate $\lambda = 0.5$ per minute, and that customers will leave immediately if either window is occupied by another customer. Model this with a continuous-time Markov chain.
6.3 Birth and Death Processes

- Consider a system in which the state at time $t$ is the number of people $n$ in the system at time $t$.

- Suppose further:
  (i) people enter the system with interarrival times that are exponential with rate $\lambda_n$.
  (ii) people depart the system with interdeparture times that are exponential with rate $\mu_n$.
  (iii) arrivals and departures are independent.
- This process $\{X(t)\}$ is a birth and death process with arrival (or birth) rates $\{\lambda_n\}$ and departure (or death) rates $\{\mu_n\}$.

- The state space of $\{X(t)\}$ is:

- What are the $\{v_j\}$ and $\{p_{ij}\}$ of this process?
- Note that a **Poisson process** is simply a birth and death process with

- A **pure birth** process is a birth and death process with

- So the Poisson process is

- Another pure birth process is the **Yule process**; it has

- The Yule process arises if each of $n$ individuals has exponential birth rate $\lambda$, making the birth rate of the total population _____.