6.4 The Transition Probability Function

- In Chapter 4, we studied the n-step transition probabilities $P_{ij}^{(n)}$ in a discrete-time Markov chain.

Defn. The transition probability function of a continuous-time Markov chain is

- This is the probability that a chain that is currently in state i will, t time units later, be in state j.

Example (Pure birth process): Consider a pure birth process with birth rates $\{\lambda_n\}$. If the process is in state i, let X_i be the time it spends in i before transitioning to state $i+1$. Thus

Note $X(t) < j \Leftrightarrow$

For $i<j$,

Since $X_1, \dots, X_{j-1}$ are

Example (Yule Process): If $\lambda_n = n\lambda$ for $n \geq 1$, it can be shown (see Example #6.8) that

- So in a Yule process with initial population size 1, the population size at time t is:

- If the initial population size is i individuals, then the total population size at time t is the ____ of ________ r.v.'s, which is:

Transition Probabilities for General Chains

Defn. Let the instanteous transition rate be the rate that a chain in state i makes a transition into state j:

- Note that

and

so knowing the $\{q_{ij}\}$ values is enough to know all the $\{v_i\}$ and $\{P_{ij}\}$ values of the process.

<u>Lemma 1</u>: (a) $\lim_{h \to 0} \frac{1 - P_{ii}(h)}{h} = v_i$

and (b) $\lim_{h \to 0} \frac{P_{ij}(h)}{h} = q_{ij}$ for $i \neq j$.

<u>Proof</u>:

Lemma 2 (Chapman-Kolmogorov Equations):

For $s \geq 0$, $t \geq 0$,

$$P_{ij}(t+s) = \sum_{k=0}^{\infty} P_{ik}(t)\, P_{kj}(s)$$

Proof:

Theorem: (Kolmogorov's Backward Equations)

For all i, j and $t \geq 0$:

$$P_{ij}'(t) = \sum_{k \neq i} q_{ik}\, P_{kj}(t) - v_i\, P_{ij}(t)$$

Proof:

Special Cases: (1) For a pure birth process,

(2) For a birth and death process,

<u>Example</u> (A Single Machine): Suppose a machine works for an amount of time that is exponential with mean 50 days. When it fails, it must be repaired, and the repair time is exponential with mean 2.5 days. If the machine is working at the beginning of January, what is the probability it will be working at the end of January?

- Then we have a

Let $h(t) =$

Theorem (Kolmogorov's Forward Equations):

For continuous-time Markov chains (including all finite-state models and all birth and death processes) that meet certain regularity conditions,

$$P_{ij}'(t) = \sum_{k \neq j} q_{kj} P_{ik}(t) - v_j P_{ij}(t)$$

Proof:

The rest of the proof is nearly identical to the proof of the Backward Equations. Only difference: The interchange of limit and summation does not always hold.

Special case: For a birth and death process, the Forward Equations are: