Chapter 1  #20  There are \(6^3 = 216\) possible outcomes, all equally likely. There are 5 outcomes like \((1,1,x)\) where \(x \neq 1\); 5 outcomes like \((1, x, 1)\); and 5 outcomes like \((x, 1, 1)\). That is 15 outcomes that correspond to the event of interest. Also, the same argument holds for outcomes like \((2,2,x)\), \((2,x,2)\), \(\ldots\), \((x, 6, 6)\). So altogether \((15)(6) = 90\) outcomes correspond to the event of interest. So \(P[\text{exactly 2 dice match}] = \frac{90}{216} = \frac{5}{12} \approx 0.417\).

#21  Let \(C = \text{person is color-blind}\), \(M = \text{person is male}\), \(F (=\bar{M}) = \text{Person is Female}\). By Bayes' Formula:

\[
P(M|C) = \frac{P(C|M)P(M)}{P(C|M)P(M) + P(C|F)P(F)}
\]

\[
= \frac{(0.05)(0.5)}{(0.05)(0.5) + (0.0025)(0.5)} \approx 0.9524
\]

#33  Let \(n\) be the number of sophomore girls, let event \(F = \text{"choose a freshman"}\), and let \(B = \text{"choose a boy"}\). Then \(P(F) = \frac{4+6}{4+6+6+n} = \frac{10}{16+n}\) and \(P(F|B) = \frac{4}{10}\). "Sex" and "Class" are independent if and only if \(P(F) = P(F|B)\), so we need \(\frac{10}{16+n} = \frac{4}{10} \Rightarrow 100 = 64 + 4n \Rightarrow 36 = 4n \Rightarrow n = 9\).
Let \( H = \) heads, \( T = \) tails, \( W = \) white ball selected.

By Bayes' Formula:

\[
P(T | W) = \frac{P(W | T) P(T)}{P(W | T) P(T) + P(W | H) P(H)}
\]

\[
= \frac{(\frac{3}{15})(0.5)}{(\frac{3}{15})(0.5) + (\frac{5}{12})(0.5)} \approx 0.324
\]

Chapter 2 (3) If \( n = 2 \), \( X \) could be \(-2, 0, \) or \( 2 \).

\( P(X = -2) = \frac{1}{4}, \ P(X = 0) = \frac{1}{2}, \ P(X = 2) = \frac{1}{4}. \)

(26) a) The possible outcomes (labeled by game winners) are: \( AA, BB, ABA, BAA, ABB, BAB. \) The first 2 correspond to \( X = 2 \) and the rest to \( X = 3 \). The first outcome has probability \( p^2 \), the second \( (1-p)^2 \), the next two \( p^2 (1-p) \) and the last two \( p(1-p)^2 \). So \( E(X) = \sum_x x P(x) \)

\[
= 2[p^2 + (1-p)^2] + 3[2p^2(1-p) + 2p(1-p)^2]
\]

\[
\]

\[
= 2 - 4p + 4p^2 + 6p^2 - 6p^3 + 6p - 12p^2 + 6p^3
\]

\[
= 2 + 2p - 2p^2. \quad \text{\( E(X) \) maximized when} \ \frac{d}{dp}[E(x)] = 0 \Rightarrow 2 - 4p = 0 \Rightarrow p = \frac{1}{2}
\]

(Calculations could be simplified a bit by noting that the outcomes can be defined simply based on the first Ttwo Games.)
28 a) Let the outcomes as: \( A = (O_1 = T, O_2 = T) \)
\( B = (O_1 = H, O_2 = H) \), \( C = (O_1 = T, O_2 = H) \), \( D = (O_1 = H, O_2 = T) \).
Note that outcomes A and B can be disregarded since they reset the process. Then the event "\( X = 0 \)"
is the same as: \( C \mid (C \text{ or } D) \)
\[
\Rightarrow P(X=0) = \frac{P(C \text{ and } (C \text{ or } D))}{P(C \text{ or } D)} = \frac{P(C)}{P(C \text{ or } D)}
\]
\[
= \frac{(1-p)p}{(1-p)p + p(1-p)} = \frac{(1-p)p}{2(1-p)p} = \frac{1}{2}
\]
Similarly, \( P(X=1) = \frac{P(D)}{P(C \text{ or } D)} = \frac{1}{2} \).

b) Note the final flip is a head if and only if the first flip is a tail. So
\[
P(X=0) = P(\text{first flip is tail}) = 1 - p \neq \frac{1}{2}.
\]

43 a) \( X = \sum_{i=1}^{n} X_i \)
b) \( E(X) = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} P[X_i = 1] \)
Now, "\( X_i = 1 \)" is the event that the red ball labeled \( i \) is taken before any of the \( m \) black balls. So
\[
P(X_i = 1) = \frac{1}{m+1}.
\]
So \( E(X) = \sum_{i=1}^{n} \frac{1}{m+1} = \frac{n}{m+1} \).
Chapter 3 #37 Let \( Y = \begin{cases} 2.6 & \text{with probability } \frac{1}{3} \\ 3 & \text{with probability } \frac{1}{3} \\ 3.4 & \text{with probability } \frac{1}{3} \end{cases} \)

Then \( X|Y \sim \text{Poisson}(Y) \Rightarrow E(X|Y) = Y = \text{var}(X|Y) \)

a) \( E(X) = E[E(X|Y)] = E(Y) = \frac{1}{3}(2.6) + \frac{1}{3}(3) + \frac{1}{3}(3.4) = 3 \)

b) \( \text{Var}(X) = \text{Var}[E(X|Y)] + E[\text{var}(X|Y)] = \text{Var}(Y) + E(Y) = \frac{5}{3} E(Y^2) - [E(Y)]^2 + 3 = \frac{5}{3} (2.6)^2 + \frac{1}{3}(3)^2 + \frac{1}{3}(3.4)^2 - 9 \frac{7}{3} + 3 = 9.1067 - 9 + 3 = 3.1067 \)

#38 \( Y \sim \text{Unif}(0,1) \) and \( X|Y \sim \text{Unif}(0, Y) \)

\( E(X) = E[E(X|Y)] = E\left[\frac{Y}{2}\right] = \frac{1}{2} E(Y) = \frac{1}{2} \left(\frac{1}{2}\right) = \frac{1}{4} \)

\( \text{Var}(X) = \text{Var}[E(X|Y)] + E[\text{var}(X|Y)] = \text{var}\left[\frac{Y}{2}\right] + E\left[\frac{Y^2}{12}\right] \)

\( = \frac{1}{4} \text{Var}(Y) + \frac{1}{12} E(Y^2) = \frac{1}{4} \left(\frac{1}{12}\right) + \frac{1}{12} \left\{ \text{var}(Y) + [E(Y)]^2 \right\} \)

\( = \frac{1}{48} + \frac{1}{12} \left\{ \frac{1}{12} + \frac{1}{4} \right\} = 0.0486 \)

Graduate Student Problems:

Chapter 1 #18 Outcomes are: \( \{BB, BG, GB, GG\} \).

Let events \( E = \text{eldest is girl}, \ F = \text{at least one is girl}, \ D = \text{both are girls} \). Then \( E = \{GB, GG\}, \quad F = \{BG, GB, GG\}, \quad D = \{GG\} \).

a) \( P(D|E) = \frac{P(DE)}{P(E)} = \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{2} \) since \( DE = \{GG\} \)
Chapter 2 #1 ERR, RO, RB, OB, OO, BB. If balls are selected simultaneously. Not all of these are equally likely. X could be 0, 1, 2. There are \( ^{10}C_2 \) ways to select 2 balls from 10, and \( ^{7}C_2 \) ways to select two balls from the 7 non-orange balls. So

\[
P(X=0) = P(\text{select two non-orange balls}) = \frac{\binom{7}{2}}{\binom{10}{2}} = 0.4667.
\]

Chapter 3 #36 a) Note that

\[
E(X) = E(X|X \neq 0)P(X \neq 0) + E(X|X = 0)P(X = 0)
\]

\[
\Rightarrow \mu = E(X|X \neq 0)(1-po) + [0] po
\]

\[
\Rightarrow E(X|X \neq 0) = \frac{\mu}{1-po}
\]

b) \[\text{var}(X|X \neq 0) = E(X^2|X \neq 0) - [E(X|X \neq 0)]^2\]

\[
E(X^2) = \text{var}(X) + [E(X)]^2 = \sigma^2 + \mu^2
\]

\[
\Rightarrow \sigma^2 + \mu^2 = E(X^2|X \neq 0)(1-po) + E(X^2|X = 0)P(X = 0)
\]

\[
\Rightarrow E(X^2|X \neq 0) = \frac{\sigma^2 + \mu^2}{1-po} - \frac{\mu^2}{(1-po)^2}
\]

\[
\Rightarrow \text{var}(X|X \neq 0) = \frac{\sigma^2 + \mu^2}{1-po} - \frac{\mu^2}{(1-po)^2}
\]
Let $\mathbb{A} = \text{the event that } "A \text{ wins}"$. Let $X$ be the overall number of games played. Let $Y$ be the number of games won by $A$ out of the first two games.

Then $P(A) = P(A \mid Y = 0) P(Y = 0) + P(A \mid Y = 1) P(Y = 1) + P(A \mid Y = 2) P(Y = 2) = \binom{0}{0} (1-p)^2 + \binom{1}{0} P(A)(2p(1-p)) + \binom{2}{0} p^2$

since if $Y = 1$, the series "resets".

$\Rightarrow P(A) = P(A) [2p(1-p)] + p^2$

$P(A) = P(A) 2p - 2p^2 P(A) + p^2$

$P(A) [1 - 2p + 2p^2] = p^2 \Rightarrow P(A) = \frac{p^2}{1 - 2p + 2p^2}$

$E(X) = E(X \mid Y = 0) P(Y = 0) + E(X \mid Y = 1) P(Y = 1) + E(X \mid Y = 2) P(Y = 2) = \binom{2}{0} (1-p)^2 + \binom{2}{1} E(X)(2p(1-p)) + \binom{2}{2} p^2$

$\Rightarrow E(X) = 2(1-p)^2 + 4p(1-p) + [E(X)]2p(1-p) + 2p^2$

$\Rightarrow E(X) [1 - 2p(1-p)] = 2(1-p)^2 + 4p(1-p) + 2p^2$

$\Rightarrow E(X) = \frac{2(1-p)^2 + 4p(1-p) + 2p^2}{1 - 2p(1-p)}$