STAT 521 HW 1 Example Solutions

#1 There are $6^3 = 216$ possible outcomes, all equally likely. There are 5 outcomes like $(1,1,x)$ where $x \neq 1$; 5 outcomes like $(1, x, 1)$; and 5 outcomes like $(x,1,1)$. That is 15 outcomes that correspond to the event of interest. Also, the same argument holds for outcomes like $(2,2,x)$, $(2,x,2)$, $\ldots$, $(x,6,6)$. So altogether $(15)(6) = 90$ outcomes correspond to the event of interest.

So $P[\text{exactly 2 dice match}] = \frac{90}{216} = \frac{5}{12} \approx 0.417$.

#2. Let C = person is color-blind, M = person is male, F $(= M^c)$ = Person is Female. By Bayes' Formula:

$$P(M \mid C) = \frac{P(C \mid M)\,P(M)}{P(C \mid M)P(M) + P(C \mid F)\,P(F)}$$

$$= \frac{(.05)(.5)}{(.05)(.5) + (.0025)(.5)} \approx 0.9524$$

#3 Let n be the number of sophomore girls, let event F = "choose a freshman", and let B = "choose a boy". Then $P(F) = \frac{4+6}{4+6+6+n} = \frac{10}{16+n}$ and $P(F \mid B) = \frac{4}{10}$. "Sex" and "Class" are independent if and only if $P(F) = P(F \mid B)$, so we need $\frac{10}{16+n} = \frac{4}{10}$

$\Rightarrow 100 = 64 + 4n \Rightarrow 36 = 4n \Rightarrow \boxed{n = 9}$

#4 Let H = heads, T = tails, W = white ball selected.
By Bayes' Formula:

$$P(T|W) = \frac{P(W|T)\,P(T)}{P(W|T)\,P(T) + P(W|H)\,P(H)}$$

$$= \frac{\left(\frac{3}{15}\right)(0.5)}{\left(\frac{3}{15}\right)(0.5) + \left(\frac{5}{12}\right)(0.5)} \approx \boxed{0.324}$$

#5

Outcome		X_1	X_2	$X = X_1 - X_2$
TT	$\Longrightarrow$	0	2	-2
TH	$\Longrightarrow$	1	1	0
HT	$\Longrightarrow$	1	1	0
HH	$\Longrightarrow$	2	0	2

Since each of the four outcomes has probability $1/4$,

$P(X=-2) = \frac{1}{4}$, $P(X=0) = \frac{1}{2}$, $P(X=2) = \frac{1}{4}$.

#6 a) The possible outcomes (labeled by game winners) are: AA, BB, ABA, BAA, ABB, BAB. The first 2 correspond to $X=2$ and the rest to $X=3$. The first outcome has probability p^2, the second $(1-p)^2$, the next two $p^2(1-p)$ and the last two $p(1-p)^2$. So $E(X) = \sum_x x\,p(x)$

$$= 2\left[p^2 + (1-p)^2\right] + 3\left[2p^2(1-p) + 2p(1-p)^2\right]$$
$$= 2\left[1 - 2p + 2p^2\right] + 3\left[2p^2 - 2p^3 + 2p - 4p^2 + 2p^3\right]$$
$$= 2 - 4p + 4p^2 + 6p^2 - 6p^3 + 6p - 12p^2 + 6p^3$$
$$= 2 + 2p - 2p^2.$$

$E(X)$ maximized when $\frac{d}{dp}[E(X)] = 0 \Rightarrow 2 - 4p = 0 \Rightarrow p = \frac{1}{2}$

(Calculations could be simplified a bit by noting that the ~~[illegible]~~ outcomes can be defined simply based on the first TWO GAMES)

#7 a) $X = \sum_{i=1}^{n} X_i$

b) $E(X) = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} P[X_i = 1]$

Now, "$X_i = 1$" is the event that the red ball labeled i is taken before any of the m black balls. So $P(X_i = 1) = \frac{1}{m+1}$. So $E(X) = \sum_{i=1}^{n} \frac{1}{m+1} = \boxed{\frac{n}{m+1}}$.

#8) Let $Y = \begin{cases} 2.6 & \text{with probability } 1/3 \\ 3 & \text{with probability } 1/3 \\ 3.4 & \text{with probability } 1/3 \end{cases}$

Then $X|Y \sim \text{Poisson}(Y) \Rightarrow E(X|Y) = Y = Var(X|Y)$

a) $E(X) = E[E(X|Y)] = E(Y) = \frac{1}{3}(2.6) + \frac{1}{3}(3) + \frac{1}{3}(3.4)$
$= 3.$

b) $Var(X) = Var[E(X|Y)] + E[var(X|Y)] = Var(Y) + E(Y)$
$= \{E(Y^2) - [E(Y)]^2\} + 3 = \{\frac{1}{3}(2.6)^2 + \frac{1}{3}(3)^2 + \frac{1}{3}(3.4)^2$
$- 9\} + 3 = 9.1067 - 9 + 3 = \boxed{3.1067}$

#9) $Y \sim Unif(0,1)$ and $X|Y \sim Unif(0, Y)$

$E(X) = E[E(X|Y)] = E\left[\frac{Y}{2}\right] = \frac{1}{2}E(Y) = \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4}$

$Var(X) = var[E(X|Y)] + E[var(X|Y)] = var\left[\frac{Y}{2}\right] + E\left[\frac{Y^2}{12}\right]$
$= \frac{1}{4}var(Y) + \frac{1}{12}E(Y^2) = \frac{1}{4}\left(\frac{1}{12}\right) + \frac{1}{12}\{var(Y) + [E(Y)]^2\}$
$= \frac{1}{48} + \frac{1}{12}\{\frac{1}{12} + \frac{1}{4}\} = 0.0486$

Graduate Student Problems:

#10) Outcomes are: $\{BB, BG, GB, GG\}$.

Let events E = Eldest is girl, F = at least one is girl, D = both are girls. Then $E = \{GB, GG\}$, $F = \{BG, GB, GG\}$, $D = \{GG\}$

a) $P(D|E) = \frac{P(DE)}{P(E)} = \frac{1/4}{1/2} = \frac{1}{2}$ since $DE = \{GG\}$

b) $P(D|F) = \dfrac{P(DF)}{P(F)} = \dfrac{1/4}{3/4} = 1/3$ since $DF = \{GG\}$

#11 $\{RR, RO, RB, OB, OO, BB\}$ if balls are selected simultaneously. Not all of these are equally likely. X could be 0, 1, 2. There are $\binom{10}{2}$ ways to select 2 balls from 10, and $\binom{7}{2}$ ways to select two balls from the 7 non-orange balls. So

$P(X=0) = P(\text{select two non-orange balls})$

$$= \frac{\binom{7}{2}}{\binom{10}{2}} = 0.4667.$$

#12 a) Note that

$$E(X) = E(X|X\neq 0)P(X\neq 0) + E(X|X=0)P(X=0)$$

$$\Rightarrow \mu = E(X|X\neq 0)(1-p_0) + [0]\, p_0$$

$$\Rightarrow E(X|X\neq 0) = \boxed{\frac{\mu}{1-p_0}}$$

b) $var(X|X\neq 0) = E(X^2|X\neq 0) - [E(X|X\neq 0)]^2$

$E(X^2) = var(X) + [E(X)]^2 = \sigma^2 + \mu^2$

$$E(X^2) = E(X^2|X\neq 0)P(X\neq 0) + E(X^2|X=0)P(X=0)$$

$$\Rightarrow \sigma^2 + \mu^2 = E(X^2|X\neq 0)(1-p_0) + [0]\, p_0$$

$$\Rightarrow E(X^2|X\neq 0) = \frac{\sigma^2+\mu^2}{1-p_0}$$

$$\Rightarrow var(X|X\neq 0) = \frac{\sigma^2+\mu^2}{1-p_0} - \frac{\mu^2}{(1-p_0)^2}$$

#13 Let A = the event that "A wins".
Let X be the overall number of games played.
Let Y be the number of games won by A out of the first two games.

Then $P(A) = P(A \mid Y=0)P(Y=0) + P(A \mid Y=1)P(Y=1)$
$+ P(A \mid Y=2)P(Y=2)$

$$= [0][(1-p)^2] + [P(A)][2p(1-p)] + [1]p^2$$

since if $Y=1$, the ~~game~~ series "resets".

$$\Rightarrow P(A) = P(A)[2p(1-p)] + p^2$$

$$P(A) = P(A)2p - 2p^2 P(A) + p^2$$

$$P(A)[1 - 2p + 2p^2] = p^2 \Rightarrow \boxed{P(A) = \frac{p^2}{1-2p+2p^2}}$$

$$E(X) = E(X \mid Y=0)P(Y=0) + E(X \mid Y=1)P(Y=1) + E(X \mid Y=2)P(Y=2)$$

$$= [2][(1-p)^2] + [2 + E(X)][2p(1-p)] + [2][p^2]$$

$$\Rightarrow E(X) = 2(1-p)^2 + 4p(1-p) + [E(X)]2p(1-p) + 2p^2$$

$$\Rightarrow E(X)[1 - 2p(1-p)] = 2(1-p)^2 + 4p(1-p) + 2p^2$$

$$\Rightarrow \boxed{E(X) = \frac{2(1-p)^2 + 4p(1-p) + 2p^2}{1 - 2p(1-p)}}$$

which simplifies to $\boxed{E(X) = \frac{2}{1-2p(1-p)}}$