#5 To find $E[X_3]$, we need to derive the unconditional probability distribution of $X_3$. The initial probabilities are $\alpha_0 = \frac{1}{4}$, $\alpha_1 = \frac{1}{4}$, $\alpha_2 = \frac{1}{2}$. Then note that

$$P^3 = \begin{bmatrix}
0.3611 & 0.2037 & 0.4352 \\
0.4444 & 0.1481 & 0.4074 \\
0.4167 & 0.2222 & 0.3611
\end{bmatrix}$$

So $P[X_3 = 0] = \alpha_0 P_{00}^3 + \alpha_1 P_{10}^3 + \alpha_2 P_{20}^3$

$$= (\frac{1}{4})(0.3611) + (\frac{1}{4})(0.4444) + (\frac{1}{2})(0.4167) = 0.4097$$

$P[X_3 = 1] = \alpha_0 P_{01}^3 + \alpha_1 P_{11}^3 + \alpha_2 P_{21}^3$

$$= (\frac{1}{4})(0.2037) + (\frac{1}{4})(0.1481) + (\frac{1}{2})(0.2222) = 0.1991$$

$P[X_3 = 2] = \alpha_0 P_{02}^3 + \alpha_1 P_{12}^3 + \alpha_2 P_{22}^3$

$$= (\frac{1}{4})(0.4352) + (\frac{1}{4})(0.4074) + (\frac{1}{2})(0.3611) = 0.3912$$

So $E(X_3) = (0)(0.4097) + (1)(0.1991) + (2)(0.3912) = 0.9815$

#10 Alter the Markov chain so that state 2 ("Glum") is an absorbing state, i.e., let

$$P = \begin{bmatrix}
0.5 & 0.4 & 0.1 \\
0.3 & 0.4 & 0.3 \\
0 & 0 & 1
\end{bmatrix}$$

Then $P[\text{not being glum any of the next 3 days}] = P[\text{not being in the absorbing state after 3 days} | \text{initial state is 0}] = 1 - P_{02}^3$
\[
P^3 = \begin{bmatrix}
0.293 & 0.292 & 0.415 \\
0.219 & 0.220 & 0.561 \\
0 & 0 & 1
\end{bmatrix}
\]

So \(1 - P^3_{02} = 1 - 0.415 = 0.585\)

**14** Chain with \(P_1\): class is \(\{0, 1, 2, 3\}\), which is recurrent.

Chain with \(P_2\): class is \(\{0, 1, 2, 3, 4\}\), which is recurrent.

Chain with \(P_3\): classes are \(\{0, 2, 3, 4\}\), \(\{0, 3, 4\}\). \(\{0, 2, 3\}\) is recurrent since once the chain enters that class, it is certain to reenter it (since it must remain there). Similarly, \(\{3, 4\}\) is recurrent. But \(\{1\}\) is transient since it is possible the chain could leave state 1 and never return.

Chain with \(P_4\): classes are \(\{0, 1, 3, 4\}\), \(\{2, 3\}\), \(\{3, 4\}\).

\(\{4\}\) is recurrent (absorbing state), \(\{3\}\) is transient (chain could leave state 3 and never return). \(\{4\}\) is clearly transient.

**18a** This can be considered a Markov chain with states \(\{0, 1, 2, 3\}\) and:

\[
P = \begin{bmatrix}
0.6 & 0.4 \\
0.5 & 0.5
\end{bmatrix}
\]
The long-run proportion for Coin 1 can be found by:
\[ \pi_1 = 0.6 \pi_1 + 0.5 \pi_2 \]
\[ \pi_1 + \pi_2 = 1 \]
\[ \Rightarrow \pi_2 = 1 - \pi_1 \]
\[ \Rightarrow \pi_1 = 0.6 \pi_1 + 0.5 \left( 1 - \pi_1 \right) \]
\[ \Rightarrow \pi_1 = 0.1 \pi_1 + 0.5 \Rightarrow 0.9 \pi_1 = 0.5 \Rightarrow \pi_1 = \frac{5}{9} \]

b) This is \( P_{12}^5 = 0.44444 = \frac{4}{9} \)

Since \( P^5 = \begin{bmatrix} 0.55556 & 0.44444 \\ 0.55555 & 0.44445 \end{bmatrix} \)

Let State 0 = good and state 1 = bad. Then
\[ P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \]

Note \( P(X_2 = 0 | X_0 = 0) = P_{00}^2 \)

and \( P(X_2 = 1 | X_0 = 0) = P_{01}^2 \). And \( P^2 = \begin{bmatrix} 0.4167 & 0.5833 \\ 0.3889 & 0.6111 \end{bmatrix} \)

Also \( P(X_1 = 0 | X_0 = 0) = P_{00} \) and \( P(X_1 = 1 | X_0 = 0) = P_{01} \).

Let \( T_i = \) total number of storms in year \( i \).

\[ E(T_1 + T_2) = E(T_1) + E(T_2) \]
\[ = (1)(P_{00}) + (3)(P_{01}) + (1)(P_{00}^2) + (3)(P_{01}^2) \]
\[ = (1)(\frac{1}{2}) + (3)(\frac{1}{2}) + (1)(0.4167) + (3)(0.5833) \]
\[ = 4.167 \]

b) \( P[T_3 = 0] = P[T_3 = 0 | X_3 = 0] P[X_3 = 0] \)
\[ + P[T_3 = 0 | X_3 = 1] P[X_3 = 1] \]
\[
= P[T_3 = 0 \mid T_3 \sim \text{Pois}(1)] P_0^3 + P[T_3 = 0 \mid T_3 \sim \text{Pois}(3)] P_1^3
\]
\[
= (0.3679)(0.4028) + (0.0498)(0.5972) = 0.1779
\]

\(c\) The long-run probabilities for the two states (good + bad) can be found by:

\[
\pi_0 = \frac{1}{2} \pi_0 + \frac{1}{3} \pi_1,
\]
\[
\pi_0 + \pi_1 = 1
\]

\[\Rightarrow \pi_1 = 1 - \pi_0 \Rightarrow \pi_0 = \frac{1}{2} \pi_0 + \frac{1}{3} (1 - \pi_0)
\]
\[\Rightarrow \frac{1}{2} \pi_0 = \frac{1}{3} - \frac{1}{3} \pi_0 \Rightarrow \frac{5}{6} \pi_0 = \frac{1}{3} \Rightarrow \pi_0 = 0.4
\]

and so \(\pi_1 = 0.6\)

So the long-run average number of storms per year is:

\[(1)(0.4) + (3)(0.6) = 2.2\]

\[
\text{Let state 0 = truck, state 1 = car. Then}
\]
\[
P = \begin{bmatrix}
0.1 & 0.34 \\
0.5 & 0.45
\end{bmatrix}
\]

So we want \(\pi_0\).

Solve:

\[
\pi_0 = \frac{1}{4} \pi_0 + \frac{1}{5} \pi_1,
\]
\[
\pi_0 + \pi_1 = 1
\]

\[\Rightarrow \pi_0 = \frac{1}{4} \pi_0 + \frac{1}{5} (1 - \pi_0) \Rightarrow \frac{3}{4} \pi_0 = \frac{1}{5} - \frac{1}{5} \pi_0
\]

\[\Rightarrow \frac{19}{20} \pi_0 = \frac{1}{5} \Rightarrow \pi_0 = 0.2105
\]
#36 a) \(P[\text{good on Tue}]\)

\[= P[\text{good on Tue} \mid \text{state 0 on Tue}] P[\text{state 0 on Tue}] + P[\text{good on Tue} \mid \text{state 1 on Tue}] P[\text{state 1 on Tue}]\]

\[= (p_0)(P_{00}) + (p_1)(P_{01}) = 0.4p_0 + 0.6p_1\]

b) If \(P = \begin{bmatrix} 0.4 & 0.6 \\ 0.2 & 0.8 \end{bmatrix}\) then \(P^4 = \begin{bmatrix} 0.2512 & 0.7488 \\ 0.2496 & 0.7504 \end{bmatrix}\)

So similarly as in part (a),

\[P[\text{good on Fri}] = (p_0)(P_{00}^4) + (p_1)(P_{01}^4) = 0.2512p_0 + 0.7488p_1\]

c) The long-run proportions of states 0 and 1 are found by:

\[\pi_0 = 0.4\pi_0 + 0.2\pi_1, \quad \pi_0 + \pi_1 = 1\]

\[\Rightarrow \pi_0 = 0.4\pi_0 + 0.2(1 - \pi_0)\]

\[\Rightarrow 0.6\pi_0 = 0.2 = 0.2\pi_0 \Rightarrow 0.8\pi_0 = 0.2\]

\[\Rightarrow \pi_0 = \frac{1}{4}, \quad \pi_1 = \frac{3}{4}\]

So long-run proportion of good messages is:

\[\frac{1}{4}p_0 + \frac{3}{4}p_1\]