STAT 521 - HW 2 Example Solutions

#1 To find $E[X_3]$, we need to derive the unconditional probability distribution of X_3.
The initial probabilities are $\alpha_0 = \frac{1}{4}$, $\alpha_1 = 1/4$, $\alpha_2 = \frac{1}{2}$. Then note that

$$P^3 = \begin{bmatrix} .3611 & .2037 & .4352 \\ .4444 & .1481 & .4074 \\ .4167 & .2222 & .3611 \end{bmatrix}$$

So
$$P[X_3 = 0] = \alpha_0 P_{00}^3 + \alpha_1 P_{10}^3 + \alpha_2 P_{20}^3$$
$$= (\tfrac{1}{4})(.3611) + (\tfrac{1}{4})(.4444) + (\tfrac{1}{2})(.4167) = .4097$$
$$P[X_3 = 1] = \alpha_0 P_{01}^3 + \alpha_1 P_{11}^3 + \alpha_2 P_{21}^3$$
$$= (\tfrac{1}{4})(.2037) + (\tfrac{1}{4})(.1481) + (\tfrac{1}{2})(.2222) = .1991$$
$$P[X_3 = 2] = \alpha_0 P_{02}^3 + \alpha_1 P_{12}^3 + \alpha_2 P_{22}^3$$
$$= (\tfrac{1}{4})(.4352) + (\tfrac{1}{4})(.4074) + (\tfrac{1}{2})(.3611) = .3912$$

So $E(X_3) = (0)(.4097) + (1)(.1991) + (2)(.3912) = \boxed{0.9815}$

#2 Alter the Markov chain so that state 2 ("Glum") is an absorbing state, i.e., let

$$P = \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0 & 0 & 1 \end{bmatrix}$$

Then P[not being glum any of the next 3 days]

= P[not being in the absorbing state after 3 days | initial state is 0]

$= 1 - P_{02}^3$

$$P^3 = \begin{bmatrix} .293 & .292 & .415 \\ .219 & .220 & .561 \\ 0 & 0 & 1 \end{bmatrix}$$

So $1 - P^3_{02} = 1 - .415 = \boxed{0.585}$

#3 Chain with P_1: class is $\{0,1,2\}$, which is recurrent.

Chain with P_2: class is $\{0,1,2,3\}$, which is recurrent.

Chain with P_3: classes are $\{0,2\}$, $\{1\}$, $\{3,4\}$. $\{0,2\}$ is recurrent since once the chain enters that class, it is certain to reenter it (since it must remain there). Similarly, $\{3,4\}$ is recurrent. But $\{1\}$ is transient since it is possible the chain could leave state 1 and never return.

Chain with P_4: classes are $\{0,1\}$, $\{2\}$, $\{3\}$, $\{4\}$. Note $3 \to 2$ but $2 \not\to 3$. And $4 \to 0$ but $0 \not\to 4$. $\{0,1\}$ is recurrent. $\{2\}$ is recurrent (absorbing state). $\{3\}$ is transient (chain could leave state 3 and never return). $\{4\}$ is clearly transient.

#4 a) This can be considered a Markov chain with states $\{1,2\}$ and:

$$P = \begin{bmatrix} 0.6 & 0.4 \\ 0.5 & 0.5 \end{bmatrix}$$

The long-run proportion for Coin 1 can be found by:

$$\pi_1 = 0.6\,\pi_1 + 0.5\,\pi_2$$
$$\pi_1 + \pi_2 = 1$$

$\Rightarrow \pi_2 = 1 - \pi_1 \Rightarrow \pi_1 = 0.6\,\pi_1 + 0.5(1-\pi_1)$

$\Rightarrow \pi_1 = 0.1\,\pi_1 + 0.5 \Rightarrow 0.9\,\pi_1 = 0.5 \Rightarrow \boxed{\pi_1 = \frac{5}{9}}$

b) This is $P^5_{12} = .44444 = \frac{4}{9}$

since $P^5 = \begin{bmatrix} .55556 & .44444 \\ .55555 & .44445 \end{bmatrix}$

#5 Let State 0 = good and state 1 = bad. Then $P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$. Note $P(X_2=0 \mid X_0=0) = P^2_{00}$ and $P(X_2=1 \mid X_0=0) = P^2_{01}$. And $P^2 = \begin{bmatrix} .4167 & .5833 \\ .3889 & .6111 \end{bmatrix}$

Also $P(X_1=0 \mid X_0=0) = P_{00}$ and $P(X_1=1 \mid X_0=0) = P_{01}$.

Let T_i = total number of storms in year i.

$$E(T_1+T_2) = E(T_1) + E(T_2)$$
$$= (1)(P_{00}) + (3)(P_{01}) + (1)(P^2_{00}) + (3)(P^2_{01})$$
$$= (1)\left(\tfrac{1}{2}\right) + (3)\left(\tfrac{1}{2}\right) + (1)(.4167) + (3)(.5833)$$
$$= \boxed{4.167}$$

b) $P[T_3=0] = P[T_3=0 \mid X_3=0]\,P[X_3=0] + P[T_3=0 \mid X_3=1]\;P[X_3=1]$

$$= P[T_3 = 0 \mid T_3 \sim \text{Pois}(1)]\, P^3_{00} + P[T_3 = 0 \mid T_3 \sim \text{Pois}(3)]\, P^3_{01}$$

Note

$$P^3 = \begin{bmatrix} .4028 & .5972 \\ .3981 & .6019 \end{bmatrix}$$

$$= (.3679)(.4028) + (.0498)(.5972) = \boxed{.1779}$$

c) The long-run probabilities for the two states (good + bad) can be found by:

$$\pi_0 = \tfrac{1}{2}\pi_0 + \tfrac{1}{3}\pi_1$$
$$\pi_0 + \pi_1 = 1$$

$$\Rightarrow \pi_1 = 1 - \pi_0 \Rightarrow \pi_0 = \tfrac{1}{2}\pi_0 + \tfrac{1}{3}(1 - \pi_0)$$

$$\Rightarrow \tfrac{1}{2}\pi_0 = \tfrac{1}{3} - \tfrac{1}{3}\pi_0 \Rightarrow \tfrac{5}{6}\pi_0 = \tfrac{1}{3} \Rightarrow \pi_0 = 0.4$$

and so $\pi_1 = 0.6$

So the long-run average number of storms per year is: $(1)(0.4) + (3)(0.6) = \boxed{2.2}$

#6 Let state 0 = truck, state 1 = car. Then

$$P = \begin{bmatrix} 1/4 & 3/4 \\ 1/5 & 4/5 \end{bmatrix}.$$ So we want π_0.

Solve:

$$\pi_0 = \tfrac{1}{4}\pi_0 + \tfrac{1}{5}\pi_1$$
$$\pi_0 + \pi_1 = 1$$

$$\Rightarrow \pi_0 = \tfrac{1}{4}\pi_0 + \tfrac{1}{5}(1 - \pi_0) \Rightarrow \tfrac{3}{4}\pi_0 = \tfrac{1}{5} - \tfrac{1}{5}\pi_0$$

$$\Rightarrow \tfrac{19}{20}\pi_0 = \tfrac{1}{5} \Rightarrow \boxed{\pi_0 = .2105}$$

#7 a) $P[\text{good on Tue}]$

$= P[\text{good on Tue} \mid \text{state 0 on Tue}]\, P[\text{state 0 on Tue}]$

$+ P[\text{good on Tue} \mid \text{state 1 on Tue}]\, P[\text{state 1 on Tue}]$

$= (p_0)(P_{0,0}) + (p_1)(P_{0,1}) = \boxed{0.4\, p_0 + 0.6\, p_1}$

b) If $P = \begin{bmatrix} 0.4 & 0.6 \\ 0.2 & 0.8 \end{bmatrix}$ then $P^4 = \begin{bmatrix} .2512 & .7488 \\ .2496 & .7504 \end{bmatrix}$

So similarly as in part (a),

$P[\text{good on Fri}] = (p_0)(P_{00}^4) + (p_1)(P_{01}^4)$

$= \boxed{.2512\, p_0 + .7488\, p_1}$

c) The long-run proportions of states 0 and 1 are found by:

$$\pi_0 = 0.4\,\pi_0 + 0.2\,\pi_1$$
$$\pi_0 + \pi_1 = 1$$

$\Rightarrow \pi_0 = 0.4\,\pi_0 + 0.2(1 - \pi_0)$

$\Rightarrow 0.6\,\pi_0 = 0.2 - 0.2\,\pi_0 \Rightarrow 0.8\,\pi_0 = 0.2$

$\Rightarrow \pi_0 = \frac{1}{4}, \quad \pi_1 = \frac{3}{4}.$

So long-run proportion of good messages is:

$$\boxed{\frac{1}{4} p_0 + \frac{3}{4} p_1}$$