STAT 521 - HW 2 Example Solutions

1. To find $E[X_3]$, we need to derive the unconditional probability distribution of $X_3$.

The initial probabilities are $\alpha_0 = \frac{1}{4}$, $\alpha_1 = \frac{1}{4}$, $\alpha_2 = \frac{1}{2}$. Then note that

$$P^3 = \begin{bmatrix} .3611 & .2037 & .4352 \\ .4444 & .1481 & .4074 \\ .4167 & .2222 & .3611 \end{bmatrix}$$

So

$$P[X_3 = 0] = \alpha_0 P_{00}^3 + \alpha_1 P_{10}^3 + \alpha_2 P_{20}^3$$

$$= \left(\frac{1}{4}\right)(.3611) + \left(\frac{1}{4}\right)(.4444) + \left(\frac{1}{2}\right)(.4167) = .4097$$

$$P[X_3 = 1] = \alpha_0 P_{01}^3 + \alpha_1 P_{11}^3 + \alpha_2 P_{21}^3$$

$$= \left(\frac{1}{4}\right)(.2037) + \left(\frac{1}{4}\right)(.1481) + \left(\frac{1}{2}\right)(.2222) = .1991$$

$$P[X_3 = 2] = \alpha_0 P_{02}^3 + \alpha_1 P_{12}^3 + \alpha_2 P_{22}^3$$

$$= \left(\frac{1}{4}\right)(.4352) + \left(\frac{1}{4}\right)(.4074) + \left(\frac{1}{2}\right)(.3611) = .3912$$

So

$$E(X_3) = (0)(.4097) + (1)(.1991) + (2)(.3912) = \boxed{0.9815}$$

2. Alter the Markov chain so that state 2 ("Glum") is an absorbing state, i.e., let

$$P = \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0 & 0 & 1 \end{bmatrix}$$

Then

$$P[\text{not being glum any of the next 3 days}]$$

$$= P[\text{not being in the absorbing state after 3 days} | \text{initial state is 0}]$$

$$= 1 - P_{02}^3$$
\[ P^3 = \begin{bmatrix} 0.293 & 0.292 & 0.415 \\ 0.219 & 0.220 & 0.561 \\ 0 & 0 & 1 \end{bmatrix} \]

So \( 1 - P_{02}^3 = 1 - 0.415 = 0.585 \)

\#3 Chain with \( P_1 \): class is \( \{0,1,2,3\} \), which is recurrent.

Chain with \( P_2 \): class is \( \{0,1,2,3\} \), which is recurrent.

Chain with \( P_3 \): classes are \( \{0,2,3\}, \{1,3\}, \{3,4\} \). \( \{0,2,3\} \) is recurrent since once the chain enters that class, it is certain to reenter it (since it must remain there). Similarly, \( \{3,4\} \) is recurrent. But \( \{1,3\} \) is transient since it is possible the chain could leave state 1 and never return.

Chain with \( P_4 \): classes are \( \{0,1,3\}, \{2,3\}, \{3,3\}, \{4,3\} \). Note 3 \( \rightarrow \) 2 but 2 \( \not\rightarrow \) 3. And 4 \( \rightarrow \) 0 but 0 \( \not\rightarrow \) 4. \( \{0,1,3\} \) is recurrent. \( \{2,3\} \) is recurrent (absorbing state). \( \{3,3\} \) is transient (chain could leave state 3 and never return). \( \{4,3\} \) is clearly transient.

\#4a) This can be considered a Markov chain with states \( \{1,2,3\} \) and:

\[ P = \begin{bmatrix} 0.6 & 0.4 \\ 0.5 & 0.5 \end{bmatrix} \]
The long-run proportion for Coin 1 can be found by:

\[ \pi_1 = 0.6 \pi_1 + 0.5 \pi_2 \]

\[ \pi_1 + \pi_2 = 1 \]

\[ \Rightarrow \pi_2 = 1 - \pi_1 \Rightarrow \pi_1 = 0.6 \pi_1 + 0.5 (1 - \pi_1) \]

\[ \Rightarrow \pi_1 = 0.1 \pi_1 + 0.5 \Rightarrow 0.9 \pi_1 = 0.5 \Rightarrow \pi_1 = \frac{5}{9} \]

b) This is \[ P_{12}^5 = 0.44444 = \frac{4}{9} \]

since \[ P_5^5 = \begin{bmatrix} .595556 & 0.44444 \\ 0.59555 & 0.4445 \end{bmatrix} \]

#5 Let state 0 = good and state 1 = bad. Then

\[ P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \]

Note \[ P(X_2 = 0 | X_0 = 0) = P_{00}^2 \]

and \[ P(X_2 = 1 | X_0 = 0) = P_{01}^2 \]. And \[ P^2 = \begin{bmatrix} .41667 & .58333 \\ .38889 & .61111 \end{bmatrix} \]

Also \[ P(X_1 = 0 | X_0 = 0) = P_{00} \] and \[ P(X_1 = 1 | X_0 = 0) = P_{01} \].

Let \( T_i \) = total number of storms in year \( i \).

\[ E(T_1 + T_2) = E(T_1) + E(T_2) \]

\[ = (1)(P_{00}) + (3)(P_{01}) + (1)(P_{00}^2) + (3)(P_{01}^2) \]

\[ = (1)(\frac{1}{2}) + (3)(\frac{1}{2}) + (1)(0.4167) + (3)(0.5833) \]

\[ = \boxed{4.167} \]

b) \[ P[T_3 = 0] = P[T_3 = 0 | X_3 = 0] P[X_3 = 0] + P[T_3 = 0 | X_3 = 1] P[X_3 = 1] \]
\[
\begin{align*}
= & \ P\left[ T_3 = 0 \mid T_3 \sim \text{Pois}(1) \right] P_{00}^3 \\
+ & \ P\left[ T_3 = 0 \mid T_3 \sim \text{Pois}(3) \right] P_{01}^3 \\
= & \ (0.3679)(0.4028) + (0.0498)(0.5972) = 0.1779
\end{align*}
\]

\(c\) The long-run probabilities for the two states (good + bad) can be found by:

\[
\begin{align*}
\Pi_0 &= \frac{1}{2} \Pi_0 + \frac{1}{3} \Pi_1 \\
\Pi_0 + \Pi_1 &= 1
\end{align*}
\]

\[\Rightarrow \ Pi_1 = 1 - \Pi_0 \Rightarrow \ Pi_0 = \frac{1}{2} \Pi_0 + \frac{1}{3} \left( 1 - \Pi_0 \right)\]

\[\Rightarrow \ \frac{1}{2} \Pi_0 = \frac{1}{3} - \frac{1}{3} \Pi_0 \Rightarrow \frac{5}{6} \Pi_0 = \frac{1}{3} \Rightarrow \Pi_0 = 0.4\]

and so \(\Pi_1 = 0.6\)

\(\Rightarrow\) So the long-run average number of storms per year is: \((1)(0.4) + (3)(0.6) = 2.2\)

\(\#6\) Let state 0 = truck, state 1 = car. Then

\[
P = \begin{bmatrix}
\frac{1}{4} & \frac{3}{4} \\
\frac{1}{5} & \frac{4}{5}
\end{bmatrix}
\]

So we want \(\Pi_0\).

Solve:

\[
\begin{align*}
\Pi_0 &= \frac{1}{4} \Pi_0 + \frac{1}{5} \Pi_1 \\
\Pi_0 + \Pi_1 &= 1
\end{align*}
\]

\[\Rightarrow \Pi_0 = \frac{1}{4} \Pi_0 + \frac{1}{5} \left( 1 - \Pi_0 \right) \Rightarrow \frac{3}{4} \Pi_0 = \frac{1}{5} - \frac{1}{5} \Pi_0 \]

\[\Rightarrow \frac{19}{20} \Pi_0 = \frac{1}{5} \Rightarrow \Pi_0 = 0.2105\]
\(\#7\) a) \(P[\text{good on Tue}]\)

\[= P[\text{good on Tue} | \text{state 0 on Tue}] P[\text{state 0 on Tue}] + P[\text{good on Tue} | \text{state 1 on Tue}] P[\text{state 1 on Tue}]\]

\[= (p_0)(P_{0,0}) + (p_1)(P_{0,1}) = 0.4p_0 + 0.6p_1\]

b) If \(P = \begin{bmatrix} 0.4 & 0.6 \\ 0.2 & 0.8 \end{bmatrix}\) then \(P^4 = \begin{bmatrix} 0.2512 & 0.7488 \\ 0.2496 & 0.7504 \end{bmatrix}\)

So similarly as in part (a),

\[P[\text{good on Fri}] = (p_0)(P_{0,0}^4) + (p_1)(P_{0,1}^4)\]

\[= 0.2512p_0 + 0.7488p_1\]

c) The long-run proportions of states 0 and 1 are found by:

\[\pi_0 = 0.4\pi_0 + 0.2\pi_1\]

\[\pi_0 + \pi_1 = 1\]

\[\Rightarrow \pi_0 = 0.4\pi_0 + 0.2(1 - \pi_0)\]

\[\Rightarrow 0.6\pi_0 = 0.2 - 0.2\pi_0 \Rightarrow 0.8\pi_0 = 0.2\]

\[\Rightarrow \pi_0 = \frac{1}{4}, \quad \pi_1 = \frac{3}{4}\]

So long-run proportion of good messages is:

\[\frac{1}{4}p_0 + \frac{3}{4}p_1\]