STAT 521 HW 3 Example Solutions

#1 This is just the Gambler's ruin problem with $i = m$ and goal $N = n+m$. Plugging into the formula for the probability of reaching the goal, we have

$$P[\text{reach goal}] = \frac{1-\left(\frac{q}{p}\right)^m}{1-\left(\frac{q}{p}\right)^{n+m}} \quad \text{if} \quad p \neq \tfrac{1}{2}$$

where $q = 1-p$, and $= \dfrac{m}{n+m}$ if $p = \frac{1}{2}$.

#2 The transient states are $\{1,2,3\}$, so

$$P_T = \begin{bmatrix} 0.4 & 0.2 & 0.1 \\ 0.1 & 0.5 & 0.2 \\ 0.3 & 0.4 & 0.2 \end{bmatrix}$$

$$S = (I - P_T)^{-1} = \begin{bmatrix} 2.207 & 1.379 & 0.621 \\ 0.966 & 3.103 & 0.897 \\ 1.310 & 2.069 & 1.931 \end{bmatrix}$$

So $s_{13} = 0.621$, $s_{23} = 0.897$, $s_{33} = 1.931$

$$f_{13} = \frac{s_{13} - 0}{s_{33}} = \frac{0.621}{1.931} = 0.322$$

$$f_{23} = \frac{s_{23} - 0}{s_{33}} = \frac{0.897}{1.931} = 0.465$$

$$f_{33} = \frac{s_{33} - 1}{s_{33}} = \frac{0.931}{1.931} = 0.482$$

(#3) We want $E\left[\sum_{n=0}^{\infty} X_n \mid X_0 = 1\right]$

$$= \sum_{n=0}^{\infty} E[X_n \mid X_0 = 1] = \sum_{n=0}^{\infty} \mu^n = \frac{1}{1-\mu} \text{ if } \mu < 1.$$

If $X_0 = n$, consider n branches, each starting with 1 individual. The expected number for each branch is $\frac{1}{1-\mu}$, so the expected number overall is $\frac{n}{1-\mu}$.

(#4) a) $\mu = (0)(\frac{1}{4}) + (2)(\frac{3}{4}) = 1.5$

Set $\pi_0 = 0.25 + 0.75\,\pi_0^2$. The smallest positive solution of this is $\frac{1}{3}$, so $\pi_0 = \frac{1}{3}$.

b) $\mu = (0)(\frac{1}{4}) + (1)(\frac{1}{2}) + (2)(\frac{1}{4}) = 1$, so $\pi_0 = 1$.

c) $\mu = (0)(\frac{1}{6}) + (1)(\frac{1}{2}) + (2)(\frac{1}{3}) = 1.167$

Set $\pi_0 = \frac{1}{6} + \frac{1}{2}\pi_0 + \frac{1}{3}\pi_0^2$. The smallest positive solution of this is 0.5, so $\pi_0 = 0.5$.

(5) a) Write $P_i = P[\text{enter state 2 before state 3} \mid \text{start in } i]$

So $P_0 = P[\text{enter 2 before 3} \mid X_0 = 0]$

$= P[\text{enter 2 before 3} \mid X_0=0, X_1=0]\,P[X_1=0 \mid X_0=0]$

$+ P[\text{enter 2 before 3} \mid X_0=0, X_1=1]\,P[X_1=1 \mid X_0=0]$

$+ P[\text{enter 2 before 3} \mid X_0=0, X_1=2]\,P[X_1=2 \mid X_0=0]$

$+ P[\text{enter 2 before 3} \mid X_0=0, X_1=3]\,P[X_1=3 \mid X_0=0]$

$\Rightarrow P_0 = (P_0)(0.4) + (P_1)(0.2) + (1)(0.1) + (0)(0.3)$

$\Rightarrow P_0 = 0.4P_0 + 0.2P_1 + 0.1 \qquad (*)$

Also, $P_1 = P[\text{enter 2 before 3} \mid X_0 = 1]$

$= P[\text{enter 2 before 3} \mid X_0=1, X_1=0]\,P[X_1=0 \mid X_0=1]$

$+ P[\text{enter 2 before 3} \mid X_0=1, X_1=1]\,P[X_1=1 \mid X_0=1]$

$+ P[\text{enter 2 before 3} \mid X_0=1, X_1=2]\,P[X_1=2 \mid X_0=1]$

$+ P[\text{enter 2 before 3} \mid X_0=1, X_1=3]\,P[X_1=3 \mid X_0=1]$

$\Rightarrow P_1 = (P_0)(0.2) + (P_1)(0.2) + (1)(0.1) + (0)(0.5)$

$\Rightarrow P_1 = 0.2P_0 + 0.2P_1 + 0.1 \Rightarrow 0.8P_1 = 0.2P_0 + 0.1$

$\Rightarrow P_1 = 0.25P_0 + 0.125$. Substituting back into $(*)$:

$P_0 = 0.4P_0 + 0.2(0.25P_0 + 0.125) + 0.1$

$\Rightarrow 0.55P_0 = 0.125 \Rightarrow \boxed{P_0 \approx 0.2273 \text{ or } \frac{5}{22}}$

⑤ b) Consider making "states 2 and 3" an absorbing state so that we have a Markov chain with transition probability matrix

$$\begin{bmatrix} 0.4 & 0.2 & 0.4 \\ 0.2 & 0.2 & 0.6 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that states 0 and 1 are transient and the absorbing state is recurrent. So

$$P_T = \begin{bmatrix} 0.4 & 0.2 \\ 0.2 & 0.2 \end{bmatrix} \quad \text{and} \quad S = (I - P_T)^{-1}$$

is $S = \begin{bmatrix} 1.8182 & 0.4545 \\ 0.4545 & 1.3636 \end{bmatrix}$. So starting from state 0, the mean time spent in transient states 0 and 1 before entering the absorbing state is:

$$S_{0,0} + S_{0,1} = 1.8182 + 0.4545$$
$$= \boxed{2.2727}$$