STAT 521 HW 5 Example Solutions

#1 Let T_1 = time until first kidney comes
T_2 = time between first and second kidney arrivals
W_A = A's survival time without kidney
W_B = B's survival time without kidney

$T_1 \sim \text{expon}(\lambda)$, $T_2 \sim \text{expon}(\lambda)$, $W_A \sim \text{expon}(\mu_A)$, $W_B \sim \text{expon}(\mu_B)$
and all these are independent.

(a) $P[\text{A obtains kidney}] = P[T_1 < W_A] = \boxed{\dfrac{\lambda}{\lambda + \mu_A}}$

(b) $P[\text{B obtains kidney}]$

$= P$["B survives until A gone" AND "B survives until next kidney arrives"]

(where "A gone" means "A dies or gets a kidney, whichever comes first")

$= P[\text{B survives until A gone}] \cdot$
$P[\text{B survives until next kidney} \mid \text{B survives until A gone}]$

$$= P[\min\{W_A, T_1\} < W_B] \, P[T_i < W_B]$$

whichever $i = 1, 2$ is appropriate once A gone

$$= \left(\frac{\lambda + \mu_A}{\lambda + \mu_A + \mu_B}\right)\left(\frac{\lambda}{\lambda + \mu_B}\right)$$

since $\min\{W_A, T_1\} \sim \text{expon}(\lambda + \mu_A)$

(c) $P[A \text{ and } B \text{ die before first kidney}]$

$= P[A \text{ dies}]\, P[B \text{ dies} \mid A \text{ dies}]$

$= P[T_1 > W_A]\, P[T_1 > W_B]$

$= \left(\dfrac{\mu_A}{\lambda + \mu_A}\right)\left(\dfrac{\mu_B}{\lambda + \mu_B}\right)$

(d) $P[A \text{ and } B \text{ both get kidneys}]$

$= P[A \text{ and } B \text{ survive until first kidney arrives}] \cdot$

$P[B \text{ survives until second kidney} \mid A \text{ and } B \text{ survive until first kidney}]$

$= P[A \text{ survive until first kidney}]\, P[B \text{ survive until first kidney}]$

$\cdot P[B \text{ survive until second kidney} \mid A \text{ and } B \text{ survive until first}]$

(by independence of W_A and W_B)

$= P[T_1 < W_A]\, P[T_1 < W_B]\, P[T_2 < W_B]$

$= \left(\dfrac{\lambda}{\lambda + \mu_A}\right)\left(\dfrac{\lambda}{\lambda + \mu_B}\right)\left(\dfrac{\lambda}{\lambda + \mu_B}\right) = \boxed{\dfrac{\lambda^3}{(\lambda+\mu_A)(\lambda+\mu_B)^2}}$

\# 2 a) $E[S_4] = \dfrac{4}{\lambda}$ since $S_4 \sim \text{gamma}(4, \lambda)$

b) $E[S_4 \mid N(1) = 2] = E[S_2] + 1 = \dfrac{2}{\lambda} + 1$

c) $E[N(4) - N(2) \mid N(1) = 3] = E[N(4) - N(2)]$ (by indep. increments)

$= 2\lambda$, since $N(4) - N(2) \sim \text{Pois}(2\lambda)$.

#3 a) $P[\text{win}] = P[N(T) - N(s) = 1]$

$= P[N(T-s) = 1]$ (by stationary increments)

$$= \frac{e^{-\lambda(T-s)}[\lambda(T-s)]^1}{1!} = e^{-\lambda T} e^{\lambda s}[\lambda(T-s)]$$

b) Note $P[\text{win}]$ is greatest when $e^{\lambda s}(T-s)$ is maximized $\Leftrightarrow$ $\lambda s + \ln(T-s)$ is maximized

$\Leftrightarrow \lambda + \frac{-1}{T-s} = 0 \Leftrightarrow \lambda + \frac{1}{s-T} = 0$

$\Leftrightarrow \lambda(s-T) + 1 = 0 \Leftrightarrow \lambda s = \lambda T - 1 \Leftrightarrow s = \frac{\lambda T - 1}{\lambda}$

or $s = T - \frac{1}{\lambda}$

c) Plug in s from part (b) into $P[\text{win}]$:

$$e^{-\lambda T} e^{\lambda T - 1}\left[\lambda\left(T - T + \tfrac{1}{\lambda}\right)\right] = e^{-1}\left[\lambda\left(\tfrac{1}{\lambda}\right)\right] = \frac{1}{e}$$

4 a) $E(X) = \int_0^1 E(X|Y=y)\, f_Y(y)\, dy$ where $Y \sim \text{Unif}(0,1)$

$$= \int_0^1 7y\,[1]\,dy = \left[\frac{7y^2}{2}\right]_0^1 = \frac{7}{2} = \boxed{3.5}$$

b) $E(X^2) = \int_0^1 E(X^2|Y=y) f_Y(y)\, dy = \int_0^1 [(7y)^2 + 7y]\, dy$

$$= \left[\frac{49y^3}{3} + \frac{7y^2}{2}\right]_0^1 = \frac{49}{3} + \frac{7}{2} = 19.83$$

$\Rightarrow \text{var}(X) = 19.83 - (3.5)^2 = \boxed{7.58}$

NOTE: For parts (a) and (b), you can also directly use the formulas $E(X) = E[E(X|Y)]$ and $\text{var}(X) = E[\text{var}(X|Y)] + \text{var}[E(X|Y)]$

(5) $\{N_1(t)\}$ and $\{N_2(t)\}$ are Poisson processes with rates $\lambda p = 10$ and $\lambda(1-p) = 1$. So

$$\lambda = \frac{10}{p} \Rightarrow \frac{10}{p}(1-p) = 1 \Rightarrow \frac{10}{p} - 10 = 1 \Rightarrow \frac{10}{p} = 11$$

$$\Rightarrow p = \frac{10}{11} \quad \text{so } P[\text{Type I}] = \frac{10}{11},\ P[\text{Type II}] = \frac{1}{11}.$$

Assuming only whole-dollar claims are possible, the probability that the claim is \$4000 is approximately equal to the exponential pdf evaluated at 4000. So:

$P[\text{Type I} \mid C = 4000]$

$$= \frac{P[C=4000 \mid \text{Type I}]\ P[\text{Type I}]}{P[C=4000 \mid \text{Type I}]\,P[\text{Type I}] + P[C=4000 \mid \text{Type II}]\,P[\text{Type II}]}$$

$$= \frac{\frac{1}{1000} e^{-\frac{4000}{1000}} \left(\frac{10}{11}\right)}{\frac{1}{1000} e^{-\frac{4000}{1000}} \left(\frac{10}{11}\right) + \frac{1}{5000} e^{-\frac{4000}{5000}} \left(\frac{1}{11}\right)}$$

$$= \frac{e^{-4}\left(\frac{10}{11}\right)}{e^{-4}\left(\frac{10}{11}\right) + \frac{1}{5} e^{-4/5}\left(\frac{1}{11}\right)} = \boxed{0.671}$$