(1) This is a nonhomogeneous Poisson process with intensity function

$$\lambda(t) = \begin{cases} 4, & 0 \le t \le 2 \\ 8 & 2 < t \le 4 \\ t+4 & 4 \le t < 6 \\ -2t+22 & 6 \le t < 9 \end{cases}$$

where $t =$ # hours past 8:00 a.m.

So the number of customers entering in the whole 9-hour day is $N(9)$, which is a Poisson r.v. with mean $m(9) - m(0)$, or

$$\int_0^9 \lambda(y)\,dy$$

$$= \int_0^2 4\,dy + \int_2^4 8\,dy + \int_4^6 (y+4)\,dy + \int_6^9 (-2y+22)\,dy$$

$$= [4y]_0^2 + [8y]_2^4 + \left[\frac{y^2}{2} + 4y\right]_4^6 + \left[-y^2 + 22y\right]_6^9$$

$$= 8 + (32-16) + \left[\left(\frac{36}{2}+24\right) - \left(\frac{16}{2}+16\right)\right] + \left[(-81+198) - (-36+132)\right]$$

$= 63$. So the distribution of $N(9)$ is Poisson(63), i.e., with pmf $\dfrac{e^{-63}(63)^n}{n!}$.

(2) The number of claims paid over 1 week is Poisson(5). $X(t)$, the amount paid in t weeks, is a compound Poisson r.v.

So $E[X(4)] = (5)(4)(2000) = \boxed{40000}$, since $E(Y_1) = 2000$.

$\text{var}[X(4)] = (5)(4)E[Y_1^2] = (20)\left[(E(Y_1))^2 + \text{var}(Y_1)\right]$

$= 20\left[2000^2 + 2000^2\right] = \boxed{160{,}000{,}000}$

3 $\lambda_i = (i+1)\lambda$, $i \geq 0$, and $\mu_i = i\mu$, $i \geq 0$

so for $\lambda = 2$, $\mu = 1.5$:

$\lambda_0 = 2$, $\lambda_1 = 4$, $\lambda_2 = 6$, $\lambda_3 = 8$, $\lambda_4 = 10, \ldots$

$\mu_0 = 0$, $\mu_1 = 1.5$, $\mu_2 = 3$, $\mu_3 = 4.5$, $\mu_4 = 6, \ldots$

(a) We want $E[T_0] + E[T_1] + E[T_2] + E[T_3]$

$$E(T_0) = \frac{1}{2} = 0.5$$

$$E(T_1) = \frac{1}{\lambda_1} + \frac{\mu_1}{\lambda_1} E(T_0) = \frac{1}{4} + \frac{1.5}{4}(0.5) = 0.4375$$

$$E(T_2) = \frac{1}{\lambda_2} + \frac{\mu_2}{\lambda_2} E(T_1) = \frac{1}{6} + \frac{3}{6}(0.4375)$$

$$= 0.38542$$

$$E(T_3) = \frac{1}{\lambda_3} + \frac{\mu_3}{\lambda_3} E(T_2) = \frac{1}{8} + \frac{4.5}{8}(0.38542)$$

$$= 0.34180$$

So the answer is:

$$0.5 + 0.4375 + 0.38542 + 0.34180 = \boxed{1.66472}$$

(b) We want $E(T_2) + E(T_3) + E(T_4)$

Note $E(T_4) = \frac{1}{\lambda_4} + \frac{\mu_4}{\lambda_4} E(T_3)$

$$= \frac{1}{10} + \frac{6}{10}(0.34180) = 0.30508$$

So the answer is:

$$0.38542 + 0.34180 + 0.30508 = \boxed{1.0323}$$

(4) a) The state space of $\{X(t)\}$ is $\{0,1,2\}$.

$V_0 = 2\lambda$, $V_1 = \lambda + \mu$, $V_2 = \mu$. $P_{01} = 1$, $P_{02} = 0$,

$P_{10} = \frac{\mu}{\mu+\lambda}$, $P_{12} = \frac{\lambda}{\mu+\lambda}$, $P_{20} = 0$, $P_{21} = 1$

$\Rightarrow q_{01} = 2\lambda$, $q_{10} = \mu$, $q_{12} = \lambda$, $q_{21} = \mu$

(b) The Backward Equations are

$$P_{ij}'(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - v_i P_{ij}(t)$$

So the full set of Backward Equations is:

$$P_{00}'(t) = 2\lambda P_{10}(t) - 2\lambda P_{00}(t)$$
$$P_{01}'(t) = 2\lambda P_{11}(t) - 2\lambda P_{01}(t)$$
$$P_{02}'(t) = 2\lambda P_{12}(t) - 2\lambda P_{02}(t)$$

$$P_{10}'(t) = \mu P_{00}(t) + \lambda P_{20}(t) - (\lambda+\mu) P_{10}(t)$$
$$P_{11}'(t) = \mu P_{01}(t) + \lambda P_{21}(t) - (\lambda+\mu) P_{11}(t)$$
$$P_{12}'(t) = \mu P_{02}(t) + \lambda P_{22}(t) - (\lambda+\mu) P_{12}(t)$$

$$P_{20}'(t) = \mu P_{10}(t) - \mu P_{20}(t)$$
$$P_{21}'(t) = \mu P_{11}(t) - \mu P_{21}(t)$$
$$P_{22}'(t) = \mu P_{12}(t) - \mu P_{22}(t)$$

(5) a) $v_0 = \frac{1}{5} = 0.2$, $v_1 = \frac{1}{4} = 0.25$,

$v_2 = 1$, $v_3 = \frac{1}{2} = 0.5$

$P_{01} = 1$, $P_{12} = 0.3$, $P_{13} = 0.7$, $P_{20} = 1$, $P_{30} = 1$,

all other $P_{ij} = 0$.

(b) Note $q_{ij} = v_i P_{ij}$ for all i, j.

$q_{01} = (0.2)(1) = 0.2$

$q_{02} = 0$

$q_{03} = 0$

$q_{10} = 0$

$q_{12} = (0.25)(0.3) = 0.075$

$q_{13} = (0.25)(0.7) = 0.175$

$q_{20} = (1)(1) = 1$

$q_{21} = 0$

$q_{23} = 0$

$q_{30} = (0.5)(1) = 0.5$

$q_{31} = 0$

$q_{32} = 0$