1) This is a birth and death process with
$\lambda_0 = 3$, $\lambda_1 = 3$, $\lambda_n = 0$ for $n \geq 2$
$\mu_1 = 4$, $\mu_2 = 4$, $\mu_n = 0$ for $n \geq 3$

So the balance equations are

$$\lambda_0 P_0 = \mu_1 P_1 \Rightarrow 3P_0 = 4P_1$$
$$\lambda_1 P_1 = \mu_2 P_2 \Rightarrow 3P_1 = 4P_2$$
$$\text{and} \quad P_0 + P_1 + P_2 = 1$$

$$\Rightarrow 3(1-P_1-P_2) = 4P_1 \Rightarrow 3 - 3P_1 - 3P_2 = 4P_1$$
$$\Rightarrow 3 - 3P_2 = 7P_1 \Rightarrow \frac{3}{7} - \frac{3}{7}P_2 = P_1$$
$$\Rightarrow 3\left(\frac{3}{7} - \frac{3}{7}P_2\right) = 4P_2 \Rightarrow \frac{9}{7} - \frac{9}{7}P_2 = 4P_2$$
$$\Rightarrow \frac{9}{7} = \frac{37}{7}P_2 \Rightarrow \boxed{\frac{9}{37} = P_2} \Rightarrow 3P_1 = \frac{36}{37} \Rightarrow \boxed{P_1 = \frac{12}{37}}$$
$$\Rightarrow P_0 = 1 - \frac{12}{37} - \frac{9}{37} \Rightarrow \boxed{P_0 = \frac{16}{37}}$$

So $E[X(t)] = 0\left(\frac{16}{37}\right) + 1\left(\frac{12}{37}\right) + 2\left(\frac{9}{37}\right) = \boxed{\frac{30}{37} \approx 0.81}$

b) $1 - P_2 = 1 - \frac{9}{37} = \boxed{\frac{28}{37}}$

c) If $\mu_1 = 8$, $\mu_2 = 8$, then the balance equations are:

$$3P_0 = 8P_1$$
$$3P_1 = 8P_2 \Rightarrow 3(1-P_1-P_2) = 8P_1 \Rightarrow 3 - 3P_2 = 11P_1$$
$$\Rightarrow \frac{3}{11} - \frac{3}{11}P_2 = P_1 \Rightarrow 3\left(\frac{3}{11} - \frac{3}{11}P_2\right) = 8P_2 \Rightarrow \frac{9}{11} - \frac{9}{11}P_2 = 8P_2$$
$$\Rightarrow \frac{9}{11} = \frac{97}{11}P_2 \Rightarrow \frac{9}{97} = P_2 \Rightarrow 1 - P_2 = \frac{88}{97}.$$

So instead of bringing in $\frac{28}{97}\lambda \approx 2.27$ customers per hour, the barber would bring in $\frac{88}{97}\lambda \approx 2.72$ customers per hour.

(2) Let $X(t) = \#$ cars in the system and note 5 minutes $= \frac{1}{12}$ hour. This is a birth and death process with $\lambda_0 = 20$, $\lambda_1 = 20$, $\lambda_2 = 20$, $\mu_1 = 12$, $\mu_2 = 12$, $\mu_3 = 12$. The balance equations are:

$$\left.\begin{aligned} 20P_0 &= 12P_1 \\ 20P_1 &= 12P_2 \\ 20P_2 &= 12P_3 \end{aligned}\right\} \Rightarrow \begin{aligned} &\tfrac{12}{20}P_1 + P_1 + P_2 + \tfrac{20}{12}P_2 = 1 \\ &\tfrac{32}{20}P_1 + \tfrac{32}{12}P_2 = 1 \end{aligned}$$

and $P_0 + P_1 + P_2 + P_3 = 1 \Rightarrow \frac{32}{12}P_2 = 1 - \frac{32}{20}P_1$

$$\Rightarrow P_2 = \frac{12}{32} - \frac{12}{20}P_1$$

$\Rightarrow 20P_1 = 12\left(\frac{12}{32} - \frac{12}{20}P_1\right) \Rightarrow 27.2\,P_1 = 4.5$

$\Rightarrow \boxed{P_1 \approx 0.16544} \Rightarrow \boxed{P_2 \approx 0.27574} \Rightarrow P_0 = \frac{12}{20}P_1$

$\Rightarrow \boxed{P_0 \approx 0.09926} \Rightarrow P_3 = \frac{20}{12}P_1 \Rightarrow \boxed{P_3 \approx 0.45956}$

a) $1 - P_0 = \boxed{0.90074}$

b) $P_3 = \boxed{0.45956}$

(3) The states are $0 =$ working, $1 =$ phase 1 repair, $2 =$ phase 2 repair, ..., $k =$ phase k repair. Then $V_0 = \lambda$, $V_1 = \mu_1, \dots, V_k = \mu_k$ and

$P_{01} = P_{12} = P_{23} = \dots = P_{k-1,k} = P_{k,0} = 1$ and

$q_{01} = \lambda$, $q_{12} = \mu_1$, $q_{23} = \mu_2, \dots, q_{k0} = \mu_k$

a) So the balance equations are as follows:

State j	Balance equation
0	$v_0 P_0 = q_{k0} P_k \Rightarrow \lambda P_0 = \mu_k P_k$
1	$v_1 P_1 = q_{01} P_0 \Rightarrow \mu_1 P_1 = \lambda P_0$
2	$v_2 P_2 = q_{12} P_1 \Rightarrow \mu_2 P_2 = \mu_1 P_1$
$\vdots$	$\vdots$
k	$v_k P_k = q_{k-1,k} P_{k-1} \Rightarrow \mu_k P_k = \mu_{k-1} P_{k-1}$

For state i, we can see

$$\mu_i P_i = \mu_{i-1} P_{i-1} = \cdots = \mu_1 P_1 = \lambda P_0$$

$$\Rightarrow P_i = \frac{\lambda}{\mu_i} P_0 \quad \text{and} \quad \sum_{i=0}^{k} P_i = 1 \Rightarrow \sum_{i=1}^{k} P_i = 1 - P_0$$

$$\Rightarrow P_0 = 1 - \sum_{i=1}^{k} \frac{\lambda}{\mu_i} P_0 \Rightarrow P_0 + P_0 \sum_{i=1}^{k} \frac{\lambda}{\mu_i} = 1$$

$$\Rightarrow P_0 \left[1 + \sum_{i=1}^{k} \frac{\lambda}{\mu_i} \right] = 1 \Rightarrow \boxed{P_0 = \frac{1}{1 + \sum_{i=1}^{k} \frac{\lambda}{\mu_i}}}$$

$$\Rightarrow \boxed{P_i = \frac{\lambda}{\mu_i} \left[\frac{1}{1 + \sum_{i=1}^{k} \frac{\lambda}{\mu_i}} \right]} \quad \text{for } i = 1, \ldots, k$$

b) This is $P_0 = \dfrac{1}{1 + \sum_{i=1}^{k} \frac{\lambda}{\mu_i}}$.

4. Let $X(t)$ = # broken machines at time t

This is a birth and death process with

$\lambda_0 = \frac{1}{10} + \frac{1}{10} + \frac{1}{10} = \frac{3}{10}$, $\lambda_1 = \frac{1}{10} + \frac{1}{10} = \frac{1}{5}$, $\lambda_2 = \frac{1}{10}$,

$\mu_1 = \frac{1}{8}$, $\mu_2 = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$, $\mu_3 = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$.

a) Balance equations are:

$$\lambda_0 P_0 = \mu_1 P_1 \Rightarrow \frac{3}{10} P_0 = \frac{1}{8} P_1$$

$$\lambda_1 P_1 = \mu_2 P_2 \Rightarrow \frac{1}{5} P_1 = \frac{1}{4} P_2$$

$$\lambda_2 P_2 = \mu_3 P_3 \Rightarrow \frac{1}{10} P_2 = \frac{1}{4} P_3$$

and $P_0 + P_1 + P_2 + P_3 = 1$

$$\Rightarrow \frac{10}{24} P_1 + P_1 + P_2 + \frac{4}{10} P_2 = 1 \Rightarrow 1.4167 P_1 + 1.4 P_2 = 1$$

$$\Rightarrow P_1 = 0.7059 - 0.9882 P_2 \quad \text{and} \quad P_1 = \frac{5}{4} P_2$$

$$\Rightarrow 0.7059 - 0.9882 P_2 = 1.25 P_2 \Rightarrow 0.7059 = 2.2382 P_2$$

$$\Rightarrow \boxed{P_2 = 0.3154} \Rightarrow \boxed{P_1 = 0.3942} \Rightarrow P_0 = \frac{10}{24} P_1$$

$$\Rightarrow \boxed{P_0 = 0.1643} \Rightarrow P_3 = 1 - P_0 - P_1 - P_2 \Rightarrow \boxed{P_3 = 0.1261}$$

So $E[X(t)] = 0 P_0 + 1 P_1 + 2 P_2 + 3 P_3$

$$= 0.3942 + 2(.3154) + 3(.1261) = \boxed{1.4033}$$

b) $P_2 + P_3 = 0.3154 + 0.1261 = \boxed{0.4415}$

5 a)

State	Balance Equation
$j=0$	$v_0 P_0 = v_1 P_{10} P_1 + v_2 P_{20} P_2 + v_3 P_{30} P_3$
	$0.2 P_0 = (0.25)(0) P_1 + (1)(1) P_2 + (0.5)(1) P_3$
	$0.2 P_0 = P_2 + 0.5 P_3$
$j=1$	$v_1 P_1 = v_0 P_{01} P_0 + v_2 P_{21} P_2 + v_3 P_{31} P_3$
	$0.25 P_1 = (0.2)(1) P_0 + (1)(0) P_2 + (0.5)(0) P_3$
	$0.25 P_1 = 0.2 P_0$
$j=2$	$v_2 P_2 = v_0 P_{02} P_0 + v_1 P_{12} P_1 + v_3 P_{32} P_3$
	$(1) P_2 = (0.2)(0) P_0 + (0.25)(0.3) P_1 + (0.5)(0) P_3$
	$P_2 = 0.075 P_1$
$j=3$	$v_3 P_3 = v_0 P_{03} P_0 + v_1 P_{13} P_1 + v_2 P_{23} P_2$
	$(0.5) P_3 = (0.2)(0) P_0 + (0.25)(0.7) P_1 + (1)(0) P_2$
	$0.5 P_3 = 0.175 P_1$

(b) Since $P_0 + P_1 + P_2 + P_3 = 1$, $P_3 = 1 - P_0 - P_1 - P_2$ and so

$0.2 P_0 = P_2 + 0.5(1 - P_0 - P_1 - P_2)$

$\Rightarrow 0.2 P_0 = 0.075 P_1 + 0.5(1 - P_0 - P_1 - 0.075 P_1)$

$\Rightarrow 0.2 P_0 = 0.075 P_1 + 0.5 - 0.5 P_0 - 0.5 P_1 - 0.0375 P_1$

$\Rightarrow 0.7 P_0 = 0.5 - .4625 P_1$. From the second balance equation, $P_0 = 1.25 P_1$

$\Rightarrow 0.7(1.25 P_1) = 0.5 - .4625 P_1 \Rightarrow 1.3375 P_1 = 0.5$

$\Rightarrow P_1 = 0.3738 \Rightarrow 0.5 P_3 = 0.175(0.3738)$

$\Rightarrow \boxed{P_3 = 0.1308}$ Also $P_2 = 0.0280$ and $P_0 = 0.4673$

(c) See R code on course web page.

STAT 521 HW 7 Example Solutions

(6) We want the conditional distribution of

$$B(s) \mid B(t_1) = A,\ B(t_2) = B$$

By stationary increments, this has the same distribution as

$$A + B(s-t_1) \mid B(0)=0,\ B(t_2-t_1) = B-A$$

By the result from Section 10.1, this is normal with mean $\boxed{A + \frac{s-t_1}{t_2-t_1}(B-A)}$

and variance $\frac{s-t_1}{t_2-t_1}\left(t_2-t_1-(s-t_1)\right)$

$$= \boxed{\frac{s-t_1}{t_2-t_1}(t_2-s)}$$

(7) a) $P[T_a < \infty] = \lim_{t\to\infty} P[T_a \le t]$

$$= \lim_{t\to\infty} \frac{2}{\sqrt{2\pi}} \int_{|a|/\sqrt{t}}^{\infty} e^{-y^2/2}\,dy = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} e^{-y^2/2}\,dy$$

$$= \frac{2}{\sqrt{2\pi}} \sqrt{2\pi} \int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\,dy = 2\,P[Z \ge 0]$$

$$= 2\left(\tfrac{1}{2}\right) = 1.$$

where $Z \sim N(0,1)$

(8) Let $X(t)$ be the change in price (up or down from the present price of b) at a time t units after the "present". Then time $t=0$ is the present, $X(0)=0$, and $\{X(t)\}$ is standard Brownian motion.

So $P[\text{do not recover purchase price}]$

$$= 1 - P[\text{stock price hits } b+c \text{ before time } t]$$

$$= 1 - P[X(t) \text{ hits } c \text{ before time } t]$$

$$= 1 - P[T_c \le t]$$

$$= 1 - \frac{2}{\sqrt{2\pi}} \int_{c/\sqrt{t}}^{\infty} e^{-y^2/2}\, dy \quad \text{since } c > 0.$$