1. Assume a population of organisms reproduces according to a branching process as follows. Organisms reproduce independently, and each organism has \( j \) offspring with probability \( P_j \) for \( j = 0, 1, 2 \), where \( P_0 = 0.1, P_1 = 0.7, P_2 = 0.2 \).

(a) Find the expected number of offspring for a random organism.

\[
\mu = (0)(0.1) + (1)(0.7) + (2)(0.2) = 1.1
\]

(b) If the initial population size is 1, then find the expected population size at the 9th generation.

\[
\mu^9 = (1.1)^9 = 2.36
\]

(c) Find the probability that the population eventually dies out. Show your work.

\[
\begin{align*}
\pi_0 &= 0.1 + 0.7 \pi_0 + 0.2 \pi_0^2 \\
0 &= 0.1 - 0.3 \pi_0 + 0.2 \pi_0^2 \\
0 &= (1 - 2 \pi_0) (1 - \pi_0) \\
\Rightarrow \pi_0 &= \frac{1}{2}, \quad \pi_0 = 1
\end{align*}
\]

\[
\Rightarrow \pi_0 = \frac{1}{2}
\]

(d) If we are told that the zero-th generation has 1 organism AND we are told that single initial organism has 3 offspring, what is the expected population size at the 3rd generation? Briefly explain.

\[
X_1 = 3 \Rightarrow 3 \text{ branches. Go two further generations to get } E(X_3) = 3(\mu^2) = 3(1.1^2) = 3.63
\]

2. Consider a continuous random variable \( X \) that has the memoryless property. Suppose that this random variable has \( E(X) = 4 \). Find \( P(X > 3 \mid X > 1.5) \). Show work.

\[
X \sim \text{expon}(0.25) \quad P(X > 3 \mid X > 1.5) = P(X > 1.5)
\]

\[
= e^{-0.25(1.5)} = e^{-0.375} = 0.687
\]
3. Suppose that three mechanics are working on cars: Frank is working on a Ford, Harry is working on a Honda, and Tonya is working on a Toyota. Suppose that the service times for the three mechanics are exponential with mean 25 minutes for Frank, exponential with mean 16 minutes for Harry, and exponential with mean 20 minutes for Tonya. And suppose all the service times are independent.

(a) What is the probability that Frank is the first of the three to finish servicing his car? Show work.

\[
P[X_F = \min\{X_F, X_H, X_T\}] = P[X_F < \min\{X_H, X_T\}] = \frac{0.04}{0.04 + 0.0625 + 0.05} = 0.2623
\]

(b) Now find the probability that Frank is the first of the three to finish servicing his car, given that Frank started working on his car 30 minutes ago, whereas Harry and Tonya just began working on their cars. Briefly explain.

Same as (a) — memoryless property.

\[
0.2623
\]

(c) The service station manager can go home as soon as at least one of the mechanics finishes with his car. What is the expected amount of the time the manager will have to wait before he can go home? Explain your answer.

Let \( W = \min\{X_F, X_H, X_T\} \).

\[W \sim \text{expon}(0.04 + 0.0625 + 0.05)\]

\[E(W) = \frac{1}{0.04 + 0.0625 + 0.05} = 6.557 \text{ minutes}\]

(d) Consider a different day at the same service station with the same mechanics with the same service time distributions. Frank comes to work at 8:00 and Tonya comes to work at 8:30. (Harry has the day off, so there are only two mechanics there.) Frank starts to work on the first customer’s car at 8:00, while the second customer (who also arrived at 8:00) waits for whoever is the next available mechanic. What is the probability that the second customer’s car is serviced by Tonya? Show work.

\[P(S_F \geq 30) = e^{-0.04(30)} = 0.3012\]

where \( S_F = \text{Frank's service time for customer 1} \).

(f) In the situation of part (e), what is the expected service time of the second customer’s car (i.e., from when it starts being serviced to when it’s done being serviced)? Show work.

\[E(S_2 | S_F < 30) P(S_F < 30) + E(S_2 | S_F \geq 30) P(S_F \geq 30) = (25)(0.6988) + (20)(0.3012) = 23.494 \text{ minutes}\]
4. Suppose we want to use the Metropolis-Hastings algorithm to sample from a Dagum distribution with pdf

\[ f(x) = \frac{2}{x} \left( \frac{x^2}{(x^2 + 1)^2} \right), x > 0 \]

(and 0 otherwise)

(a) Explain why the exponential distribution, with mean equal to the current value in the chain, would be a reasonable choice of proposal distribution.

- Continuous distribution we can sample from.
- Has the same support, \((0, \infty)\), as target distr.

(b) If we use the proposal distribution suggested in (a), give an expression for the probability of accepting the next proposed (candidate) value in the chain. Simplify it, where possible.

\[
i = \text{current}, \ j = \text{proposed} \implies q(i, j) = \frac{1}{i} e^{-\frac{j}{i}}
\]

\[
\alpha(i, j) = \min \left\{ \frac{\pi(j) \ q(j, i)}{\pi(i) \ q(i, j)}, 1 \right\}
\]

\[
= \min \left\{ \frac{\frac{2}{j} \ \frac{j^2}{(j^2 + 1)^2} \ \frac{1}{j} \ e^{-i/j}}{\frac{2}{i} \ \frac{i^2}{(i^2 + 1)^2} \ \frac{1}{i} \ e^{-j/i}}, 1 \right\}
\]

\[
\therefore = \min \left\{ \frac{(i^2 + 1)^2 \ e^{-i/j}}{(j^2 + 1)^2 \ e^{-j/i}}, 1 \right\}
\]

(c) Why would the Gibbs sampler not be an appropriate method to sample from the distribution in this problem?

The target distribution is univariate.
5. Suppose that customer arrivals at a coffee shop follow a Poisson process with rate $\lambda = 15$ per hour.

(a) The shop opens at 7:00 a.m. What is the probability of exactly 2 customers arriving between 8:00 and 8:10 a.m.? Show work.

$$10 \text{ minutes} = \frac{1}{6} \text{ hour}$$

$$N\left(\frac{1}{6}\right) \sim \text{Pois}\left(15\left(\frac{1}{6}\right)\right) = \text{Pois}\left(2.5\right)$$

$$P(N\left(\frac{1}{6}\right) = 2) = \frac{e^{-2.5} (2.5)^2}{2!} = \boxed{0.2565}$$

(b) Suppose the first customer of the day arrived at 7:08 a.m. What is the probability that the second customer of the day arrives by 7:14 a.m.? Show work.

$$P(T_2 < 0.1)$$

$$6 \text{ minutes} = 0.1 \text{ hours}$$

$$T_2 \sim \text{expon}(15)$$

$$= 1 - e^{-15(0.1)} = \boxed{0.777}$$

(c) Find the expected arrival time (i.e., in terms of the time on the clock) of the 12th customer of the day.

$$S_{12} \sim \text{gamma}(12, 15) \Rightarrow E(S_{12}) = \frac{12}{15} = 0.8 \text{ hours}$$

$$\Rightarrow 48 \text{ minutes} \Rightarrow 7:48 \text{ a.m.}$$

(d) Suppose that 60 percent of the shop’s customers only order a drink and 40 percent order some type of food (and that different customers’ decisions whether to buy food are independent). Find the probability that exactly 4 customers order food in the first hour that the shop is open. Show work.

$$N_2(t) \sim \text{Pois}\left((0.4)(15) t\right) = \text{Pois}\left(6 t\right)$$

$$N_2(1) \sim \text{Pois}(6)$$

$$\Rightarrow P[N_2(1) = 4] = \frac{e^{-6} 6^4}{4!} = \boxed{0.134}$$

(e) Suppose that the second customer of the afternoon (i.e., the second arrival past noon) arrived at 12:20 p.m. What is the probability that the first arrival of the afternoon happened between 12:10 and 12:15 p.m.?

$$\text{First arrival } T \sim \text{Unif}(0, 20)$$

$$P(10 < T < 15) = \frac{5}{20} = \boxed{0.25}$$

Extra credit [2 points]: Consider the sum of a RANDOM number of independent exponential r.v.’s. The distribution of this sum is named for which statistician (and from which country is he?)

Cox, Britain