Homework 1 – STAT 704

1. Let \(Y_1, \ldots, Y_{n_1}\) be a sample from a population with mean \(\mu_1\) and variance \(\sigma_1^2\). Let \(Y_{21}, \ldots, Y_{2n_2}\) be a sample from another population with mean \(\mu_2\) and variance \(\sigma_2^2\). Define
   \[
   \bar{Y}_1 = \frac{1}{n_1} \sum_{j=1}^{n_1} Y_{1j}, \quad \bar{Y}_2 = \frac{1}{n_2} \sum_{j=1}^{n_2} Y_{2j}.
   \]
   (a) Find \(E(\bar{Y}_1 - \bar{Y}_2)\).
   (b) If \(\bar{Y}_1\) and \(\bar{Y}_2\) are independent, find \(\text{var}(\bar{Y}_1 - \bar{Y}_2)\).
   (c) If the two populations are normal, then does \(\bar{Y}_1 - \bar{Y}_2\) have a normal distribution? Explain why or why not.

2. Suppose \(Y_1, Y_2, \ldots, Y_n\) are independent random variables with mean \(\mu\) and variance \(\sigma^2\).
   (a) Show that
   \[
   (n - 1)S^2 = \sum_{i=1}^{n} Y_i^2 - n \bar{Y}^2.
   \]
   (b) Show that \(E(S^2) = \sigma^2\). (Hint: Use the fact that \(\text{var}(Y) = E(Y^2) - [E(Y)]^2\).)

3. Let \(Y_1, Y_2, Y_3\) be independent random variables with means \(\mu_1, \mu_2, \mu_3\) and a common variance \(\sigma^2\). Define
   \[
   \bar{Y} = \frac{1}{3} \sum_{i=1}^{3} Y_i.
   \]
   (a) Find the covariance between \(Y_1 - \bar{Y}\) and \(\bar{Y}\).
   (b) Find the expected value of \((Y_1 + 2Y_2 - Y_3)^2\).

4. Let \(Y_1\) and \(Y_2\) be random variables with expected values \(\mu_1\) and \(\mu_2\) and variances \(\sigma_1^2\) and \(\sigma_2^2\).
   (a) Show that \(\text{cov}(Y_1 + Y_2, Y_1 - Y_2) = \sigma_1^2 - \sigma_2^2\).
   (b) If \(W = Y_1 + Y_2\) and \(V = Y_1 - Y_2\), then under what condition(s) can we be assured that \(W\) and \(V\) are independent random variables?