1. Let $Y_1, \ldots, Y_{n_1}$ be a sample from a population with mean $\mu_1$ and variance $\sigma_1^2$. Let $Y_{21}, \ldots, Y_{2n_2}$ be a sample from another population with mean $\mu_2$ and variance $\sigma_2^2$. Define 

\[ \bar{Y}_1 = \frac{1}{n_1} \sum_{j=1}^{n_1} Y_{1j}, \quad \bar{Y}_2 = \frac{1}{n_2} \sum_{j=1}^{n_2} Y_{2j}. \]

(a) Find $E(\bar{Y}_1 - \bar{Y}_2)$.

(b) If $\bar{Y}_1$ and $\bar{Y}_2$ are independent, find $\text{var}(\bar{Y}_1 - \bar{Y}_2)$.

(c) If the two populations are normal (and $\bar{Y}_1$ and $\bar{Y}_2$ are independent), then does $\bar{Y}_1 - \bar{Y}_2$ have a normal distribution? Explain why or why not.

2. Suppose $Y_1, Y_2, \ldots, Y_n$ are independent random variables with mean $\mu$ and variance $\sigma^2$.

(a) Show that 

\[ (n-1)S^2 = \sum_{i=1}^n Y_i^2 - n\bar{Y}^2. \]

(b) Show that $E(S^2) = \sigma^2$. (Hint: Use the fact that $\text{var}(Y) = E(Y^2) - [E(Y)]^2$.)

3. Let $Y_1, Y_2, Y_3$ be independent random variables with means $\mu_1, \mu_2, \mu_3$ and a common variance $\sigma^2$. Define 

\[ \bar{Y} = \frac{1}{3} \sum_{i=1}^3 Y_i. \]

(a) Find the covariance between $Y_1 - \bar{Y}$ and $\bar{Y}$.

(b) Find the expected value of $(Y_1 + 2Y_2 - Y_3)^2$.

4. Let $Y_1$ and $Y_2$ be random variables with expected values $\mu_1$ and $\mu_2$ and variances $\sigma_1^2$ and $\sigma_2^2$.

(a) Show that $\text{cov}(Y_1 + Y_2, Y_1 - Y_2) = \sigma_1^2 - \sigma_2^2$.

(b) If $W = Y_1 + Y_2$ and $V = Y_1 - Y_2$, then under what condition(s) can we be assured that $W$ and $V$ are independent random variables?

5. Suppose $Y_1, \ldots, Y_5$ are independent random variables, each having a normal distribution with mean 0 and variance 1. Let $\bar{Y} = (1/5) \sum_{i=1}^5 Y_i$ and let $Y_6$ be another independent observation, also having a $N(0, 1)$ distribution. Let $W = \sum_{i=1}^5 Y_i^2$ and let $U = \sum_{i=1}^5 (Y_i - \bar{Y})^2$. Give the distribution of each of the following quantities, and carefully explain/justify your answer in each case.

(a) $W$ (b) $U$ (c) $\sqrt{5}Y_6/\sqrt{W}$ (d) $2Y_6/\sqrt{U}$ (e) $20\bar{Y}^2/U$ (f) $4Y_6^2/U$ (g) $U/4Y_6^2$ (h) $\bar{Y} + Y_6$