1 a) $L(\theta | y_1, \dots, y_n) = \prod_{i=1}^{n} f(y_i | \theta)$

$$= \prod_{i=1}^{n} \frac{\theta^3}{\Gamma(3)} y_i^{3-1} e^{-\theta y_i}$$

$$= \left[\frac{\theta^3}{\Gamma(3)} y_1^2 e^{-\theta y_1} \right] \cdots \left[\frac{\theta^3}{\Gamma(3)} y_n^2 e^{-\theta y_n} \right]$$

$$= \frac{\theta^{3n}}{[\Gamma(3)]^n} \left[\prod_{i=1}^{n} y_i^2 \right] e^{-\theta \sum_{i=1}^{n} y_i}$$

(b) The parameter θ has support on $(0, \infty)$, so a gamma distribution, which has that support, is a sensible choice.

(c) We know the mean of the gamma is $\frac{s}{r}$ and the variance is $\frac{s}{r^2}$.

So set $\frac{s}{r} = 3$ and $\frac{s}{r^2} = 1^2 = 1$.

$\Rightarrow s = r^2 \Rightarrow \frac{r^2}{r} = 3 \Rightarrow r = 3 \Rightarrow s = 9$.

So choose a gamma$(9,3)$ as the prior.

$$\Rightarrow p(\theta) = \frac{3^9}{\Gamma(9)} \theta^{9-1} e^{-3\theta}$$

① d) $p(\theta \mid y_1, \dots, y_n) \propto L(\theta \mid y_1, \dots, y_n)\, p(\theta)$

$\propto \theta^{3n} e^{-\theta \Sigma y_i}\, \theta^{9-1} e^{-3\theta}$

(can ignore normalizing constants that do not depend on θ)

$= \theta^{3n+9-1} e^{-\theta(\Sigma y_i + 3)}$

This is the kernel of a gamma distribution with shape $= 3n+9$ and rate $= \Sigma y_i + 3$

$\Rightarrow$ Posterior is gamma$(3n+9,\ \Sigma y_i + 3)$

1 e) The posterior mean is

$$\frac{3n+9}{\sum y_i+3} = \frac{3n}{\sum y_i+3} + \frac{9}{\sum y_i+3}$$

$$= \frac{3n}{n\bar{y}+3} + \frac{9}{n\bar{y}+3} = \frac{3}{\bar{y}+\frac{3}{n}} + \frac{\frac{9}{n}}{\bar{y}+\frac{3}{n}}$$

$$= \left(\frac{\bar{y}}{\bar{y}+\frac{3}{n}}\right)\underbrace{\left(\frac{3}{\bar{y}}\right)}_{\text{MLE}} + \left(\frac{\frac{3}{n}}{\bar{y}+\frac{3}{n}}\right)\underbrace{\left(\frac{9}{3}\right)}_{\text{prior mean}}$$

This one was a little tricky!

2 a) $L(\theta | y_1, \dots, y_n) = \prod_{i=1}^{n} f(y_i | \theta)$

$= \prod_{i=1}^{n} \left[\theta (1-\theta)^{y_i - 1} \right]$

$= \left[\theta (1-\theta)^{y_1 - 1} \right] \cdots \left[\theta (1-\theta)^{y_n - 1} \right]$

$= \theta^n (1-\theta)^{y_1 + \cdots + y_n - n}$

$= \theta^n (1-\theta)^{\Sigma y_i - n}$

b) $p(\theta) \propto \theta^{\alpha - 1} (1-\theta)^{\beta - 1}$

$p(\theta | y_1, \dots, y_n) \propto L(\theta | y_1, \dots, y_n) \; p(\theta)$

$\propto \theta^n (1-\theta)^{\Sigma y_i - n} \; \theta^{\alpha - 1} (1-\theta)^{\beta - 1}$

$\theta^{n + \alpha - 1} (1-\theta)^{\Sigma y_i - n + \beta - 1}$

This is the kernel of a beta distribution with parameters $n + \alpha$ and $\Sigma y_i - n + \beta$.

$\Rightarrow$ Posterior is $\text{beta}(n + \alpha, \; \Sigma y_i - n + \beta)$.

c) Yes, this is conjugate since the prior is Beta and the posterior is also Beta (with different parameter values).

d) The posterior mean is

$$\frac{n+\alpha}{n+\alpha+\sum y_i - n + \beta} = \frac{n+\alpha}{\alpha + \sum y_i + \beta}$$