Stat 509 Homework 3 Solution

1. (a)
(b) 
\[ p(\gamma < 0.5) = \int_0^{0.5} 90 y^6 (1-y) \, dy \]
\[ = \int_0^{0.5} 90 y^6 \, dy - \int_0^{0.5} 90 y^7 \, dy \]
\[ = \frac{90}{7} y^7 \bigg|_0^{0.5} - \frac{90}{8} y^8 \bigg|_0^{0.5} \]
\[ = 10 \times 0.5^7 - 9 \times 0.5^8 \]
\[ = 0.0107 \]

(c) 
\[ E(\gamma) = \int_0^1 y \cdot f_{\gamma}(y) \, dy \]
\[ = \int_0^1 y \cdot 90 y^6 (1-y) \, dy \]
\[ = \int_0^1 90 y^7 \, dy - \int_0^1 90 y^8 \, dy \]
\[ = \frac{90}{8} y^8 \bigg|_0^1 - \frac{90}{9} y^9 \bigg|_0^1 \]
\[ = 9 - \frac{90}{9} \]
\[ = 0.8182 \]

\[ E(\gamma^2) = \int_0^1 y^2 \cdot f_{\gamma}(y) \, dy \]
\[ = \int_0^1 y^2 \cdot 90 y^6 (1-y) \, dy \]
\[ = \int_0^1 90 y^8 \, dy - \int_0^1 90 y^9 \, dy \]
\[ = \frac{90}{9} y^9 \bigg|_0^1 - \frac{90}{10} y^{10} \bigg|_0 \]
\[ = \frac{90}{10} - \frac{90}{12} \]
\[ = 0.6818 \]

\[ \text{Var}(\gamma) = E(\gamma^2) - [E(\gamma)]^2 \]
\[ = 0.6818 - 0.8182^2 \]
\[ = 0.0123 \]
(d) \[ \int_0^y 90 \cdot e^{(t-t)} \, dt = 0.5 \]

Solve this equation, we will get the median \( y = 0.8377 \).

2. (a)

\[ y = \text{seq}(1,6,0.01) \]
\[ \text{pdf} = 3/y^4 \]
\[ \text{plot}(y, \text{pdf, type="l", xlab="y", ylab="PDF")} \]
\[ \text{abline(h=0)} \# \text{ puts horizontal line at 0} \]

(b)

\[ y = \text{seq}(1,6,0.01) \]
\[ \text{cdf} = 1-y^{(-3)} \]
\[ \text{plot}(y, \text{cdf, type="l", xlab="y", ylab="CDF")} \]
\[ \text{abline(h=0)} \# \text{ puts horizontal line at 0} \]
(b) When \( y > 1 \)
\[
F_r(y) = \int_1^y \frac{3}{t^4} \, dt
= \left. \frac{3}{-3} t^{-3} \right|_1^y
= 1 - y^{-3}
\]

when \( y \leq 1 \)
\[
F_r(y) = 0
\]

(c) \( E(Y) = \int_1^\infty y \cdot \frac{3}{y^4} \, dy \)
\[
= \int_1^\infty 3 \cdot y^{-3} \, dy
= \left. 3 \cdot 2y^{-2} \right|_1^\infty
= 3 \cdot 2
\]
\[
\int_1^x \frac{3}{y^4} \, dy = 0.5
\]
\[
\frac{3}{3} y^{-3} \bigg|_1^x = 0.5
\]
\[-x^{-3} + 1 = 0.5
\]
\[-x^{-3} = 0.5
\]
\[x = 3\sqrt{2} = 1.2599 \quad \leftarrow \text{median}
\]

Mean is lower because the density function is skewed to the right.

(d) \( P(Y > 4) \)
\[
= \int_4^\infty \frac{3}{t^4} \, dt
= \left. \frac{3}{-3} t^{-3} \right|_4^\infty
= 4^{-3} = 0.0156
\]
\( y = \text{seq}(0,30000,0.1) \)

\( \text{pdf} = \text{dexp}(y,0.0002) \)

\( \text{plot}(y,\text{pdf}, \text{type}="l", \text{xlab}="y", \text{ylab}="\text{PDF}") \)

\( \text{abline}(h=0) \) # puts horizontal line at 0

\[ Y \sim \text{exponential}(0.0002) \]

\[ E(Y) = \lambda^{-1} = \frac{1}{0.0002} = 5000 \]

\[ \text{var}(Y) = \lambda^{-2} = \frac{1}{0.0002^2} = 25,000,000 \]
(b) \( P(Y < 6000) \)
\[
= F_Y(6000) \\
= 1 - e^{-0.002 \times 6000} \\
= 0.6988
\]
\[\text{R-code} \]
\[> \text{pexp}(6000, 0.002)\]

\( P(Y > 12,000) \)
\[
= 1 - P(Y \leq 12,000) \\
= 1 - F_Y(12,000) \\
= 1 - (1 - e^{-0.002 \times 12000}) \\
= 0.0907
\]
\[\text{R-code} \]
\[> 1 - \text{pexp}(12000, 0.002)\]

(c) \( P(Y > y) = 0.01 \quad \Rightarrow \quad P(Y \leq y) = 0.99 \)
\[
1 - P(Y > y) = 0.99 \\
1 - F_Y(y) = 0.99 \\
1 - e^{-0.002y} = 0.99 \\
e^{-0.002y} = 0.01 \\
y = 23025.85
\]
\[\text{R-code} \]
\[> \text{qexp}(0.99, 0.002)\]

(d) From (b) we have \( P(Y < 6000) = 0.6988 \)
\[
X \sim \text{b}(25, 0.6988) \\
P(X \leq 3) \\
= 2.8457 \times 10^{-9}
\]

Each computer results in fan failure ("success") or fan not failure ("failure").

Compartets are independent.

The probability of fan failure is the same on every computer.
R-code

(b)

> pexp(6000,0.0002)
[1] 0.6988058

(c)

> 1-pexp(12000,0.0002)
[1] 0.09071795

(d)

> pbinom(3,25,0.6988)
[1] 2.845734e-09
\( y = \text{seq}(0,1000,0.01) \)
\( \text{pdf} = \text{dgamma}(y, 2.7, 0.01) \)
\( \text{plot}(y, \text{pdf}, \text{type}="l", \text{xlab}="y", \text{ylab}="PDF") \)
\( \text{abline}(h=0) \) # puts horizontal line at 0

\( Y \sim \text{Gamma} \left( 2.7, 0.01 \right) \)

\( E(Y) = \frac{\alpha}{\lambda} = \frac{2.7}{0.01} = 270 \)

\( \text{var}(Y) = \frac{\alpha \lambda}{\lambda^2} = \frac{2.7}{0.01^2} = 27000 \)

(b) \( P(Y > 365) \)

\( = 1 - P(Y \leq 365) \)

\( = 1 - F_Y(365) \)

\( = 0.2355 \)

(c) \( P(Y < y) = 0.8 \)

\( y = 390.003 \)

80% of the patients live less than 390 days.
(b)

> 1/pgamma(365,2.7,0.01)

[1] 0.2355346

(c)

> qgamma(0.8,2.7,0.01)

[1] 390.003
5. (a) Let $Y$ denote the time until the first patient is received. Then $Y \sim \text{exponential (3.5)}$

$$P(Y > 0.5)$$

$$= 1 - P(Y \leq 0.5)$$

$$= 1 - F_Y(0.5)$$

$$= 0.1738$$

R-code: \( > 1 - \text{pexp(0.5, 3.5)} \)

(b) $X \sim \text{Gamma (3, 2.5)}$

$$P(X \leq 0.25)$$

$$= F_X(0.25)$$

$$= 0.0257$$

R-code: \( > \text{pgamma(0.25, 3, 2.5)} \)

(c) The knowledge of waiting time probability helps the administrator to allocate the right amount of staff to the emergency room.
(a)

> 1 - pexp(0.5, 3.5)
[1] 0.1737739

(b)

> pgamma(0.25, 3, 2.5)
[1] 0.02565693
6.(a)

\[ y = \text{seq}(100,500,0.01) \]
\[ \text{pdf} = \text{dnorm}(y,310,45) \]
\[ \text{plot}(y, \text{pdf}, \text{type} = "l", \text{xlab} = "y", \text{ylab} = "PDF") \]
\[ \text{abline}(h=0) \] # puts horizontal line at 0

(b) \[ P(Y > 400) \]
\[ = 1 - P(Y \leq 400) \]
\[ = 0.02275 \]

\[ > \text{1-pnorm}(400,310,45) \]
\[ [1] 0.02275013 \]

(c) \[ P(Y > y) = 0.05 \]
\[ P(Y \leq y) = 1 - 0.05 = 0.95 \]
\[ y = 384.0184 \]

\[ > \text{qnorm}(0.95,310,45) \]
\[ [1] 384.0184 \]