Question 1.
(a) Define the two events

$$\begin{aligned} A &= \{\text{individual has HCV}\} \\ B &= \{\text{individual has HIV}\}. \end{aligned}$$

The first bullet says $P(A) = 0.01$. The second bullet says $P(B|A) = 0.03$. The third bullet says $P(A|B) = 0.10$.
(b) We want to compute $P(B)$. One might think to try using LOTP, but I think this would be harder. It is easiest to note that

$$0.03 = P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A|B)P(B)}{P(A)} = \frac{0.10P(B)}{0.01} = 10P(B).$$

Solving for $P(B)$, we get $P(B) = 0.003$.
(c) If A and B were independent, then $P(A|B)$ would be equal to $P(A)$. However, this is not true as stated in part (a), so A and B are dependent events.

Question 2.
(a) We envision each server as a "trial." Let Y denote the number of operational servers (out of 4). Then Y follows a binomial distribution with $n = 4$ servers, each with probability of being operational $p = 0.95$; i.e., $Y \sim b(4, 0.95)$. The probability the web service is operational is

$$\begin{aligned} P(Y \geq 3) = P(Y = 3) + P(Y = 4) &= \binom{4}{3}(0.95)^3(0.05)^1 + \binom{4}{4}(0.95)^4(0.05)^0 \\ &= 4(0.95)^3(0.05) + (0.95)^4 \approx 0.986. \end{aligned}$$

(b) We can not use the binomial distribution to answer this part because the "success" probabilities associated with each server are different. This violates one of the Bernoulli trial assumptions. The only way to do this problem is to enumerate out all of the possibilities:

$$\begin{aligned} \text{probability 3 servers are operational} &= \underbrace{(0.30)(0.80)(0.99)(0.99)}_{\text{1st server not operational}} + \underbrace{(0.70)(0.20)(0.99)(0.99)}_{\text{2nd server not operational}} \\ &+ \underbrace{(0.70)(0.80)(0.01)(0.99)}_{\text{3rd server not operational}} + \underbrace{(0.70)(0.80)(0.99)(0.01)}_{\text{4th server not operational}} \\ &\approx 0.235 + 0.137 + 0.006 + 0.006 = 0.384. \end{aligned}$$

$$\text{probability 4 servers are operational} = (0.70)(0.80)(0.99)(0.99) \approx 0.549.$$

Therefore, the probability the web service is operational is $0.384 + 0.549 = 0.933$.

Question 3.
(a) Think of each donor who visits the bank as a "trial." If Y denotes the number of donors seen to find the first O^+ donor, then Y has a geometric distribution with $p = 0.38$. Therefore, the probability the first O^+ donor will be seen on the first, second, third, or fourth donor is

$$\begin{aligned} P(Y \leq 4) &= P(Y = 1) + P(Y = 2) + P(Y = 3) + P(Y = 4) \\ &= (0.62)^0(0.38) + (0.62)^1(0.38) + (0.62)^2(0.38) + (0.62)^3(0.38) \approx 0.852. \end{aligned}$$

(b) We are waiting for the 5th "success" in this part, where "success" means that a donor has a AB^+ or AB^- blood type. The total number of donors seen follows a negative binomial distribution with $r = 5$ and $p = 0.04$.
(c) If $X \sim \text{nib}(5, 0.04)$ as in part (b), then

$$E(X) = \frac{r}{p} = \frac{5}{0.04} = 125.$$

Therefore, we would expect to have to observe 125 donors to find the 5th one with a AB^+ or AB^- blood type.

Question 4.
(a) Let Y denote the number of vehicles that arrive per hour. Then Y has a Poisson distribution with mean $\lambda = 4$ per hour. Therefore,

$$\begin{aligned} P(Y \leq 2) = P(Y = 0) + P(Y = 1) + P(Y = 2) &= \frac{4^0 e^{-4}}{0!} + \frac{4^1 e^{-4}}{1!} + \frac{4^2 e^{-4}}{2!} \\ &= e^{-4} + 4e^{-4} + 8e^{-4} \approx 0.238. \end{aligned}$$

(b) The time T until the first vehicle arrives follows an exponential distribution with $\lambda = 4$.

$$\begin{aligned} P(T > 0.5) = 1 - P(T \leq 0.5) &= 1 - F_T(0.5) \\ &= 1 - (1 - e^{-4(0.5)}) \approx 0.135. \end{aligned}$$

(c) The time until the 4th vehicle arrives follows a gamma distribution with $\alpha = 4$ and $\lambda = 4$.

Question 5.
(a) We want $P(Y < 5)$, which we can find by integrating the pdf $f_Y(y)$, that is,

$$\begin{aligned} P(Y < 5) = \int_0^5 0.02(10 - y)dy &= 0.02\left(10y - \frac{y^2}{2}\right)\Big|_0^5 \\ &= 0.02\left(50 - \frac{25}{2}\right) = 0.75. \end{aligned}$$

(b) The expected value of Y is

$$\begin{aligned} E(Y) = \int_0^{10} y \times \underbrace{0.02(10 - y)}_{\text{pdf}}\, dy &= \int_0^{10} 0.02(10y - y^2)dy \\ &= 0.02\left(5y^2 - \frac{y^3}{3}\right)\Big|_0^{10} = 0.02\left(500 - \frac{1000}{3}\right) = 3.33 \text{ tons.} \end{aligned}$$

Interpretation? Any of these are fine:

- $E(Y) = 3.33$ is the balance point of the pdf $f_Y(y)$; i.e., where the pdf would balance.
- On a given day, the expected amount of gravel sold would be 3.33 tons.
- Over many days, the average amount of gravel sold per day would be close to 3.33 tons.

Question 6.
(a) The desired probability is

$$\begin{aligned} P(100 < T < 200) &= F_T(200) - F_T(100) \\ &= \left[1 - e^{-(200/150)^{1.5}}\right] - \left[1 - e^{-(100/150)^{1.5}}\right] \\ &= e^{-(100/150)^{1.5}} - e^{-(200/150)^{1.5}} \\ &\approx 0.580 - 0.214 = 0.366. \end{aligned}$$

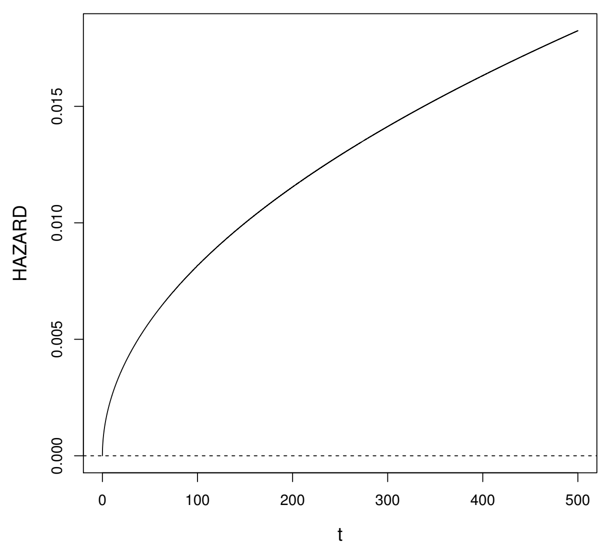
(b) We want to find the 90th percentile; i.e., the 0.9 quantile $\phi_{0.9}$. We are left to solve

$$\begin{aligned} F_T(\phi_{0.9}) &= 1 - e^{-(\phi_{0.9}/150)^{1.5}} \stackrel{\text{set}}{=} 0.9 \\ &\Longrightarrow e^{-(\phi_{0.9}/150)^{1.5}} = 0.1 \\ &\Longrightarrow -(\phi_{0.9}/150)^{1.5} = \ln(0.1) \\ &\Longrightarrow \frac{\phi_{0.9}}{150} = [-\ln(0.1)]^{1/1.5} \Longrightarrow \phi_{0.9} = 150[-\ln(0.1)]^{1/1.5} \approx 261.56. \end{aligned}$$

Therefore, because T is measured in 100s of hours, 90 percent of the resistors will fail before 26,156 hours.
(c) The Weibull hazard function is

$$h_T(t) = \frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta-1} = \frac{1.5}{150}\left(\frac{t}{150}\right)^{1.5-1} \approx 0.000816\sqrt{t}$$

A graph of this hazard function is shown below:



Because the hazard function is increasing, this means the population of resistors wears out (i.e., gets weaker) over time.