1. Suppose P_1 and P_2 are probability set functions (on a sample space S) that satisfy the Kolmolgorov Axioms. Suppose $0 \leq \alpha \leq 1$. For any $A \subset S$, define the new set function

$$P(A) = \alpha P_1(A) + (1 - \alpha) P_2(A).$$

Show that P also satisfies the Kolmolgorov Axioms.

2. A box contains $4N$ balls; $2N$ of the balls are white, $2N$ are black. The following trial is performed:

(i) $2N$ of the balls are selected at random (equally likely) and without replacement. If the event

$$A = \{\text{exactly } N \text{ of the selected balls are black}\}$$

occurs, then the experiment is stopped.

(ii) Otherwise, the $2N$ balls are replaced and the trial described in (i) is repeated.

(a) Find an expression for $P(A)$; your expression should depend on N.
(b) Find an expression for the probability that exactly y trials are required for A to occur. Assume that the trials are independent.

3. There were seven accidents in a town during a seven-day period. Define A to be the event that all seven accidents occurred on the same day. Define B to be the event that each of the seven accidents occurred on a different day. Find $P(A)$ and $P(B)$. What assumptions are you making?

4. A transmitter is sending a message in binary code ("+" and "−" signals) that must pass through two independent relay stations before being sent on to the receiver. Schematically, the message is sent as follows

$$\text{Transmitter} \implies \text{Relay 1} \implies \text{Relay 2} \implies \text{Receiver}.$$

At each relay station, there is a 25 percent chance that a signal will be reversed; that is, for $i = 1, 2$,

$$\begin{aligned} P(\text{"}+\text{" is sent by relay } i \,|\, \text{"}-\text{" is received by relay } i) &= 0.25 \\ P(\text{"}-\text{" is sent by relay } i \,|\, \text{"}+\text{" is received by relay } i) &= 0.25. \end{aligned}$$

Suppose "+" symbols make up 60 percent of the messages being sent by the transmitter. If a "+" is received from Relay 2, what is the probability a "+" was sent by the transmitter?

5. Joe and Jim each flip a fair coin n times. Find the probability that Joe and Jim flip the same number of heads.

6. A large firm is accustomed to training operators who do certain tasks on a production line. Those operators who take the training course are able to meet their production quotas 90 percent of the time. Operators who do not take the training course meet their quotas 70 percent of the time. Sixty percent of the operators have taken the training course.
(a) Find the probability an operator has taken the training course and meets his/her production quota.
(b) Find the probability an operator meets his/her production quota.
(c) Find the conditional probability an operator has taken the training course, given that s/he meets his/her production quota.
(d) Find the probability an individual has neither taken the training course nor meets his/her production quota.

7. Suppose A and B are independent events. The probability that neither occurs is a. The probability that B occurs is b.
(a) Show that

$$P(A) = \frac{1-b-a}{1-b}.$$

(b) Calculate $P(A \cup B)$.
(c) What is $P(A|B)$?

8. Suppose S is a non-empty sample space and that A and B are subsets of S with $P(A) = 0.5$, $P(B) = 0.6$, and $P(A \cup B) = 0.8$.
(a) Are A and B independent? Prove/explain your answer.
(b) Are A and B mutually exclusive? Prove/explain your answer.
(c) Find $P(\overline{A} \cup B)$.
(d) Find $P(A|B)$.
(e) Find the probability that A occurs or B occurs, but not both.

9. Clinical trials are underway to get an H1N1 (swine flu) vaccination to the public as quickly as possible. In one trial at Vanderbilt University Hospital, patients are randomly assigned to one of the four treatment groups:

1. Placebo (a non-treatment, for control purposes)
2. Experimental 1: 7.5 mcg of hemagglutinin
3. Experimental 2: 15 mcg of hemagglutinin
4. Experimental 3: 30 mcg of hemagglutinin.

There are 200 patients in the trial; 20 are randomly assigned to the Placebo group and 60 are randomly assigned to each of the experimental groups. Patients will be monitored for 6 months to see if they contract H1N1. Physicians in charge of the trial have conjectured that

- 15 percent of the Placebo patients will contract H1N1 in 6 months
- 8 percent of the Experimental 1 patients will contract H1N1 in 6 months

- 4 percent of the Experimental 2 patients will contract H1N1 in 6 months
- 1 percent of the Experimental 3 patients will contract H1N1 in 6 months

(a) Suppose a patient in the trial is selected at random. What is the probability that the patient contracts H1N1 within 6 months?
(b) Suppose a patient contracts H1N1 within 6 months. What is the probability that this patient was assigned to the Placebo group?
(c) Suppose a patient does not contract H1N1 within 6 months. What is the probability that this patient was assigned to the Experimental 3 group?

10. A project manager has 5 chemical engineers on her staff: 2 are women and 3 are men. The engineers are equally qualified. Consider the random experiment of choosing 2 engineers for 2 assignments (1 engineer per assignment).
(a) If the assignments are identical (so that order of selection is not important), write out the sample space for this experiment.
(b) Define A to be the event that no women are chosen. Under the assumption that all sample points are equally likely, find $P(A)$.
(c) For this part only, assume the 2 assignments are distinct (e.g., the assignments are in different regions of the US). Under this assumption, the sample space contains how many sample points?

11. Suppose that we have n identical white balls, numbered $1, 2, ..., n$, in one drum. Suppose that we have m identical red balls, numbered $1, 2, ..., m$, in a second drum. Consider the following experiment of

- choosing r balls from the first drum and observing the ball numbers, AND
- choosing s balls from the second drum and observing the ball numbers.

That is, balls are drawn from each drum. In addition, balls are drawn from each drum at random and without replacement; i.e., balls are not replaced after they are selected.
(a) Describe a sample space for this experiment, which, within color, does not regard the ordering of the balls drawn as important. Each sample point should be a vector of length $r + s$, corresponding to the $r + s$ numbers chosen.
(b) Suppose that we pick a sample point at random from the underlying sample space. What is the probability associated with this point? What assumptions are you making?
(c) Evaluate your expression in part (b) when $r = 5$, $n = 55$, $s = 1$, and $m = 42$. If your answer in part (b) is correct, your answer here is the probability of winning the Powerball lottery.

12. An insurance company examines its pool of auto insurance customers and gathers the following information. All customers insure at least one car; in addition,

- 70 percent of the customers insure more than one car.
- 20 percent of the customers insure a sports car.
- Of those customers who insure more than one car, 15 percent insure a sports car.

(a) Calculate the probability a randomly selected customer insures exactly one car and that car is not a sports car.
(b) Calculate the probability a randomly selected customer insures more than one car, given that s/he insures a sports car.

13. Suppose A, B, and C are events in a nonempty sample space S. Prove each of the following facts:
(a) If $P(A|B) = P(A|\overline{B})$, then A and B are independent.
(b) If $P(A|C) > P(B|C)$ and $P(A|\overline{C}) > P(B|\overline{C})$, then $P(A) > P(B)$.
(c) $P(A \cup B \cup C) \leq P(A) + P(B) + P(C)$.

14. Suppose S is a sample space and that A and B are subsets of S with $P(A) = 0.4$, $P(B) = 0.5$, and $P(A \cap B) = 0.3$. Compute each of the following probabilities:
(a) $P(\overline{A})$
(b) $P(A \cup B)$
(c) $P(\overline{A} \cap B)$
(d) $P(\overline{A} \cup \overline{B})$
(e) $P(B|A)$
(f) Are A and B independent? Why or why not?

15. I have 8 students: Jack, Jill, Fred, Fran, Clinton, Corrinne, Wes, and Wanda. I will choose 4 of these 8 students for a committee to study global warming. The committee posts are not distinct. Students are selected at random.
(a) Describe an appropriate sample space for this experiment. How many possible committees are there?
(b) Suppose that the committee chosen was Jack, Fred, Clinton, and Wes (all males; the remaining students are females). If the selection process was truly random, what is the probability of selecting this committee?
(c) How many possible committees are there if Jack and Jill will not serve together?
(d) How many possible committees are there if Clinton and Corrinne will serve together or not at all?

16. The use of plant appearance in prospecting for ore deposits is called geobotanical prospecting One indicator of copper is a small mint with a mauve-colored flower. Suppose that, for a certain region, there is a 30 percent chance that the soil has a high copper content and a 23 percent chance that the mint will be present there. In addition, we know that if the copper content is high, there is a 70 percent chance that the mint will be present. Let C denote the event that a soil sample has high copper content, and let M denote the event that the mint is present.
(a) Find the probability that the copper content will be high and the mint will be present.
(b) Find the probability that the copper content will be high given that the mint is present.

17. In one class I taught recently, there were 6 undergraduate students and 26 graduates. To investigate the studying habits of students in this course, I decided to pick a committee of 3

students from the class (at random and without replacement) to help me collect information from all 32 students. The 3 committee posts were not distinct.
(a) Suppose I wanted one undergraduate and two graduates on the committee. How many different 1-undergrad, 2-grad-committees are possible?
(b) Suppose I just randomly selected three students from the class (at random and without replacement). What is the probability there will be at least one undergraduate on the committee?

18. Suppose A and B are events in a nonempty sample space S.
(a) Show that if $P(A|B) = P(A|\overline{B})$, then A and B must be independent events. *Hint*: You might use the Law of Total Probability.
(b) Is the converse to part (a) always true? That is, if A and B are independent, does $P(A|B) = P(A|\overline{B})$ necessarily hold? Prove or give a counterexample.

19. A drawer contains 3 black, 5 green, and 2 red socks. Two socks are selected at random from the drawer. Assume that socks of the same color can not be distinguished from each other.
(a) Write out a sample space for this experiment. Are the sample points in your sample space equally likely? Explain.
(b) Compute the probability both socks are the same color.

20. A room contains 9 students: 4 are chemistry majors, 3 are business majors, and 2 are psychology majors. All of the students are lined up, at random, in the room. What is the probability the two psychology majors are lined up next to each other?

21. A track star runs two races on a certain day. Let A_1 be the event that she wins the first race and let A_2 be the event that she wins the second race. Suppose that $P(A_1) = 0.6, P(A_2) = 0.3$, and $P(A_1 \cap A_2) = 0.1$.
(a) Find the probability she wins the second race given that she wins the first race.
(b) Compute the probability she wins neither race.
(c) Are the events A_1 and A_2 independent? Justify your answer.

22. A pinball machine has 7 holes through which a ball can drop. Five balls are played and we observe which hole each ball goes down. For example, the first ball could go down hole 1, hole 2, ..., or hole 7 (similarly for the other 4 balls). On each play, assume the ball is equally likely to go down any one of the 7 holes. Find the probability that more than one ball goes down at least one of the holes.

23. Take $S = \{1, 2, ..., n\}$ and suppose that A and B, independently, are equally likely to be any of the 2^n subsets of S (including $\emptyset$ and S itself). Show that

$$P(A \subseteq B) = \left(\frac{3}{4}\right)^n.$$

Hint: Let $|B|$ denote the number of elements in B. Write $P(A \subseteq B)$ using LOTP while conditioning on $\{|B| = i\}$, for $i = 0, 1, ..., n$. Writing $A \subseteq B$ means A and B could be equal.