1. In a toxicology experiment, $Y$ denotes the death time (in minutes) for a single rat treated with a toxin. The probability density function (pdf) for $Y$ is given by

$$f_Y(y) = \begin{cases} cye^{-y/4}, & y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the value of $c$ that makes this a valid pdf.
(b) Find $P(Y < 5)$ and $P(Y \leq 5)$.
(c) Find the mean death time.

2. Suppose the random variable $Y$ has pdf

$$f_Y(y) = \begin{cases} cy, & 1 < y < 5 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the value of $c$ which makes this a valid pdf and graph this pdf.
(b) Find the cumulative distribution function $F_Y(y)$. Remember that the cdf is defined for all $-\infty < y < \infty$. Graph the cdf.
(c) Compute $E(Y)$ and $V(Y)$ without using the mgf.
(d) Derive the mgf of $Y$.

3. Explosive devices used in mining operations produce circular craters when detonated. The radii of these craters, say, $R$, follow an exponential distribution with $\beta = 10$ meters. That is, $R \sim \text{exponential}(10)$. The area of the crater is $Y = \pi R^2$.
(a) Find the mean area produced by the explosive devices; that is, compute $E(Y)$.
(b) For one detonation, find the probability the area is greater than 2500 $(\text{meters})^2$. That is, compute $P(Y > 2500)$.

4. Suppose $Y$ has the probability density function (pdf)

$$f_Y(y) = \begin{cases} 3y^2, & 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find $F_Y(y)$, the cumulative distribution function (cdf) of $Y$, and graph it. Be sure to label your axes.
(b) Find $E(Y)$.
(c) Find the 1st quartile (i.e., 25th percentile) of $Y$.

5. Suppose that $Y$ has a gamma distribution with parameters $\alpha$ and $\beta$; i.e., the pdf of $Y$ is

$$f_Y(y) = \begin{cases} \dfrac{1}{\Gamma(\alpha)\beta^\alpha} y^{\alpha-1}e^{-y/\beta}, & y > 0 \\ 0, & \text{otherwise,} \end{cases}$$

where $\alpha > 0$ and $\beta > 0$.
(a) Use the gamma moment generating function to show that $E(Y) = \alpha\beta$.
(b) For $m > 1$, show that

$$E(Y^{1/m}) = \frac{\Gamma(\alpha + 1/m)\beta^{1/m}}{\Gamma(\alpha)}.$$

6. Suppose $Y$ is a continuous random variable with probability density function (pdf)

$$f_Y(y) = \begin{cases} ky, & 0 < y < 2 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the value of $k$ and then graph the pdf.
(b) Find the cumulative distribution function (cdf) of $Y$ and graph it.
(c) Calculate $\phi_{0.9}$, the 90th percentile (0.9 quantile) of this distribution.

7. Suppose that $Y$ has a beta distribution with parameters $\alpha > 0$ and $\beta = 2$.
(a) Show that

$$E(\sqrt{Y}) = \frac{\alpha(\alpha+1)}{(\alpha+\frac{1}{2})(\alpha+\frac{3}{2})}.$$

(b) Find $\phi_{0.5}$, the median of $Y$, when $\alpha = 1$ and $\beta = 2$.

8. Let $Y$ denote the time (in years) from purchase until failure of a \$30,000 piece of manufacturing equipment. Suppose $Y$ follows an exponential distribution with mean 5.
(a) Find the probability the piece of equipment will fail before 2 years after purchase.
(b) The cost of equipment repairs ($C$, in 1000s of dollars) is a function of time:

- if the equipment fails before 1 year, the cost is $C = 0$, because the equipment is insured against failure during this time.
- if the equipment fails between 1 and 10 years, the cost is a linear function of $Y$; in particular, $C = 2Y$.
- If the equipment fails after 10 years, the cost is $C = 30$ because new equipment must be purchased.

Find the expected value of $C$.

9. The median of a continuous random variable $Y$ with cdf $F_Y(y)$ is the value $\phi_{0.5}$ such that $F_Y(\phi_{0.5}) = P(Y \leq \phi_{0.5}) = 0.5$. That is, $Y$ is just as likely to be larger than its median as it is to be smaller. For each of the distributions, calculate the median $\phi_{0.5}$.
(a) $Y \sim \mathcal{U}(0, \theta)$
(b) $Y \sim \text{exponential}(\beta)$
(c) $Y \sim \mathcal{N}(\mu, \sigma^2)$.
**Note:** For each distribution, graph the pdf and mark the median on the horizontal axis.

10. Suppose that $Y$ has a beta distribution with parameters $\alpha > 0$ and $\beta > 0$.
(a) If $\alpha = \beta$, what can be said about the probability density function (pdf) of $Y$? Prove any claims you make.
(b) Prove that

$$E(Y^k) = \frac{\Gamma(\alpha+k)\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+k)\Gamma(\alpha)}.$$

(c) Look at the result in part (b). If $k = 1$, show that the expression on the right-hand side reduces to $\alpha/(\alpha+\beta)$.

11. The performance of compressor blades in jet engines is a critical issue to engineers. Historical evidence suggests that for a particular blade, $Y$, its operational lifetime (in 100s of hours), follows a Weibull distribution with probability density function (pdf)

$$f_Y(y) = \begin{cases} y^2 e^{-y^3/3}, & 0 < y < \infty \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the cumulative distribution function (cdf) for $Y$.
(b) Compute the probability a single blade fails before 100 hours; i.e., compute $P(Y < 1)$.
(c) Derive $E(Y)$. Simplify your answer as much as possible.

12. Let $Y$ be a random variable with mean $\mu = E(Y)$ and variance $\sigma^2 = V(Y)$. Recall that the **skewness** associated with $Y$, denoted by $\xi$, is given by

$$\xi = \frac{E[(Y-\mu)^3]}{\sigma^3}.$$

The skewness $\xi$ quantifies the level of which a probability distribution departs from symmetry. Another measure that describes the distribution of $Y$ is the kurtosis. The **kurtosis** associated with $Y$, denoted by $\kappa$, is given by

$$\kappa = \frac{E[(Y-\mu)^4]}{\sigma^4}.$$

The kurtosis $\kappa$ measures the "peakedness" of a distribution; i.e., how tall the probability density function is at its peak. A normal random variable $Y$, for example, has $\kappa = 3$ irrespective of its mean or standard deviation. If a random variable's kurtosis is greater than 3, it is said to be leptokurtic. If its kurtosis is less than 3, it is said to be platykurtic. Leptokurtosis is associated with pdfs that are simultaneously "peaked" and have "fat tails." Platykurtosis is associated with pdfs that are simultaneously less peaked and have thinner tails.
(a) Derive the skewness for $Y \sim \text{gamma}(\alpha, \beta)$. Is this distribution skewed right or skewed left?
(b) Derive the kurtosis for $Y \sim \text{gamma}(\alpha, \beta)$. Is this distribution leptokurtic or platykurtic?

13. For a certain class of jet engines, the time (in years) until an overhaul is needed varies according to the following probability density function:

$$f_Y(y) = \begin{cases} ce^{-y/2}, & y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

(a) What is the value of $c$?
(b) Compute $P(Y > 2)$, and sketch a pertinent picture.
(c) Derive the cumulative distribution function of $Y$.
(d) Find the mean and variance of $T = 3Y^2 - 1$.

14. In a chemistry experiment, $Y$ denotes the velocity of a random gas molecule. According to the Maxwell-Boltzmann law, the probability density function (pdf) for $Y$ is given by

$$f_Y(y) = \begin{cases} cy^2 e^{-y^2/\beta}, & y > 0 \\ 0, & \text{otherwise,} \end{cases}$$

where $\beta > 0$ is a constant larger than zero.

(a) Find the value of $c$ that makes this a valid pdf.
(b) Write out an integral expression which equals $P(Y < 1)$. Do not evaluate the integral.

15. An environmental engineer at a large gravel company models $Y$, the monthly gravel sales (in thousands of tons) as a random variable with probability distribution function (pdf)

$$f_Y(y) = \begin{cases} 140y^3(1-y)^3, & 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Compute $E(Y)$ and $V(Y)$.
(b) Suppose that the monthly revenue (in \$100,000) from gravel sales is given by $R = 10(1-Y)$. Find the mean and standard deviation of $R$.

16. Suppose $Y$ is a continuous random variable with pdf

$$f_Y(y) = \begin{cases} 2y/\theta^2, & 0 < y < \theta \\ 0, & \text{otherwise,} \end{cases}$$

where $\theta > 0$ is a constant.
(a) Find $E(Y)$ and $V(Y)$.
(b) Find the median of $Y$.

17. During an 8-hour shift, the proportion of time $Y$ that a sheet-metal stamping machine is being serviced for repairs has the following distribution:

$$f_Y(y) = \begin{cases} 2(1-y), & 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the probability that machine repair-time is less than 30 minutes during the 8-hour shift. Sketch a pertinent picture.
(b) Find the mean and variance of $Y$.
(c) The cost $C$, measured in hundreds of dollars, of the time lost to repairs is given by $C = 10 + 20Y + 4Y^2$. Find $E(C)$ and $V(C)$.

18. Suppose that the random variable $Y$ has the following pdf

$$f_Y(y) = \begin{cases} ke^{-y^2/2}, & 0 < y < \infty \\ 0, & \text{otherwise.} \end{cases}$$

Note that $Y$ does not have a normal distribution because the support $R = \{y : y > 0\}$. In fact, $Y$ is said to have a folded-normal distribution.
(a) Argue that $k = 2/\sqrt{2\pi}$. *Hint:* For this part, use the fact that the standard normal density is a symmetric function and integrates to 1.
(b) Show that $E(Y) = 2/\sqrt{2\pi}$
(c) Show that $V(Y) = 1 - 2/\pi$.

19. Suppose that $Y \sim \mathcal{U}(\theta_1, \theta_2)$. Derive the moment generating function of $Y$ and use it to derive $E(Y)$ and $V(Y)$. Carefully explain your mgf formula when $t = 0$.

20. For a certain class of new jet engines, the time (in years) until an overhaul is needed varies according to the following probability density function:

$$f_Y(y) = \begin{cases} \frac{1}{4}e^{-y/4}, & y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Derive the cumulative distribution function (cdf) of $Y$.
(b) Find the probability that one of these engines will need an overhaul during its first year.
(c) Find $E(Y)$.
(d) Find $m_Y(t)$, the moment generating function for $Y$.

21. Suppose that $Y$ possesses the probability density function (pdf)

$$f_Y(y) = \begin{cases} y/8, & 0 < y < 4, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find $F_Y(y)$, the cumulative distribution function (cdf) of $Y$.
(b) Graph the pdf and cdf in two separate graphs, side by side. Be sure to label all axes.

22. Suppose that $Y$ is a random variable with pdf

$$f_Y(y) = \begin{cases} \theta y^{\theta-1}, & 0 < y < 1 \\ 0, & \text{otherwise,} \end{cases}$$

where $\theta > 0$.
(a) Show that $E(Y) = \theta/(\theta + 1)$.
(b) Show that the median of $Y$ is

$$m = \left(\frac{1}{2}\right)^{1/\theta}.$$

(c) It is easy to see that the mean and median will be the same when $\theta = 1$. When this is the case, what is the name of the distribution of $Y$?

23. A continuous random variable $X$ has probability density function

$$f_X(x) = \begin{cases} x, & 0 < x \le 1 \\ 2 - x, & 1 < x < 2 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Derive $F_X(x)$, the cumulative distribution function of $X$.
(b) The moment generating function (mgf) for $t \neq 0$ is

$$m_X(t) = \frac{2e^t}{t^2}(\cosh t - 1).$$

How should the mgf be defined at $t = 0$ to make $m_X(t)$ a continuous function of $t$? *Hint:* Recall from calculus that $\cosh t = (e^t + e^{-t})/2$.
(c) Find $E[(X - 1)^2]$.