GROUND RULES:

- This exam contains 8 questions; each question is worth 10 points. The maximum number of points on this exam is 80.
- This is a closed-book and closed-notes exam.
- You may use a calculator if you wish, but show all of your work and explain all of your reasoning.
- Any discussion or otherwise inappropriate communication between examinees, as well as the appearance of any unnecessary material, will be dealt with severely.
- Print your name at the top of this page in the upper right hand corner.
- You have 3 hours to complete this exam. GOOD LUCK!!

HONOR PLEDGE FOR THIS EXAM:

After you have finished the exam, please read the following statement and sign your name below it.

I promise that I did not discuss any aspect of this exam with anyone other than the instructor, that I neither gave nor received any unauthorized assistance on this exam, and that the work presented herein is entirely my own.
1. During one week, Joe will receive 7 text messages.
   (a) What is the probability that Joe will receive exactly one text message each day?
   State any assumptions you need to answer this question. I would start by characterizing
   the underlying sample space.
   (b) If 30 percent of all of Joe’s text messages are from Polly, find the probability that
   at least 2 of the 7 text messages during one week are from Polly. State any assumptions
   you need to answer this question.

2. A normalized measurement of color for automotive paint is always guaranteed to fall
   between −1 and 1. Specifically, the measurement $Y$ is a random variable with pdf
   $$f_Y(y) = \begin{cases} \frac{3}{4}(1 - y^2), & -1 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$
   (a) Find the cumulative distribution function of $Y$ and graph it.
   (b) Find the probability that $Y$ is greater than or equal to 1/2.

3. Suppose that $Y$ is a random variable with mean $\mu = E(Y)$, variance $\sigma^2 = V(Y)$, and
   moment-generating function $m_Y(t)$. Define $Z = a + bY$, where $a$ and $b$ are constants.
   (a) Show that $E(Z) = a + b\mu$.
   (b) Show that $V(Z) = b^2\sigma^2$.
   (c) Show that $m_Z(t) = e^{at}m_Y(bt)$.

4. A purchaser of electrical components buys them in lots of size 10 from two different
   suppliers. It is her policy to inspect 3 components randomly from the lot and to accept
   the lot only if all 3 are nondefective.
   
   • The purchaser buys 30 percent of her lots from Supplier 1; Supplier 1 lots always
     contain 4 defectives out of 10.
   
   • The purchaser buys 70 percent of her lots from Supplier 2; Supplier 2 lots always
     contain 1 defective out of 10.

   What is the probability that the next lot will be accepted?
   **Hint:** Define $A$ to be the event that the lot is accepted, $B_1$ to be the event that the next
   lot is from Supplier 1, and $B_2$ to be the event that the next lot is from Supplier 2. There
   are only 2 suppliers, so $\{B_1, B_2\}$ partitions the space of possible suppliers. You want to
   compute $P(A)$.

5. The lifetime $Y$ (in 1000s of hours) of a certain type of 100-watt industrial strength
   light bulb is a random variable with pdf
   $$f_Y(y) = \begin{cases} \frac{1}{2}e^{-y/2}, & 0 < y < \infty \\ 0, & \text{otherwise.} \end{cases}$$
(a) Find the probability that one randomly selected lightbulb will last longer than 3000 hours; that is, compute \( P(Y > 3) \).

(b) Suppose that we continue observing lightbulbs until we find the first one whose lifetime exceeds 3000 hours. What is the probability that we will need to observe no more than 4 lightbulbs?

(c) Suppose that we continue observing lightbulbs until we find the third one whose lifetime exceeds 3000 hours. What is the probability that we will have to observe at least 5 lightbulbs?

**Note:** In parts (b) and (c), assume that all lightbulbs are independent with the same probability from part (a).

6. Suppose that \( Y \) is a random variable with mean \( \mu \) and variance \( \sigma^2 \). Recall that the skewness and kurtosis for a random variable \( Y \) are defined as

\[
\xi = E[(Y - \mu)^3]/\sigma^3 \quad \text{(skewness)},
\]

and

\[
\kappa = E[(Y - \mu)^4]/\sigma^4 \quad \text{(kurtosis)},
\]

respectively.

(a) If \( Y \) is a **standard normal** random variable; i.e., \( Y \sim \mathcal{N}(0, 1) \), show mathematically that \( \xi = 0 \) and \( \kappa = 3 \).

(b) For any random variable, describe to me, in words, what \( \mu, \sigma^2, \xi, \) and \( \kappa \) measure. You can be brief.

7. The management at a fast-food outlet is interested in the joint behavior of the random variables \( Y_1 \) and \( Y_2 \). The variable \( Y_1 \) denotes the total time (in minutes) between a customer's arrival at the store and his/her departure from the service window. The variable \( Y_2 \) denotes the time (in minutes) a customer waits in line before reaching the service window. Both \( Y_1 \) and \( Y_2 \) are measured in minutes. The joint distribution of \( Y_1 \) and \( Y_2 \) is given by

\[
f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} 
e^{-y_1}, & 0 < y_2 < y_1 < \infty \\ 0, & \text{otherwise.} \end{cases}
\]

(a) Compute the probability that \( Y_1 - Y_2 \), the time spent at the service window, is greater than 1 minute; that is, compute \( P(Y_1 - Y_2 > 1) \).

(b) Find both marginal distributions.

(c) Find the conditional distribution of \( Y_1 \) for customers whose value of \( Y_2 = y_2 = 2.5 \).

8. Let \( Y_1 \) and \( Y_2 \) denote the proportions of two different chemicals in a mixture of chemicals used as an insecticide. Suppose that \((Y_1, Y_2)\) has the joint probability density function

\[
f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} 6y_1, & 0 \leq y_1 \leq 1, \ 0 \leq y_2 \leq 1, \ 0 \leq y_1 + y_2 \leq 1 \\ 0, & \text{otherwise.} \end{cases}
\]
(a) The set \( R = \{ (y_1, y_2) : 0 \leq y_1 \leq 1, \ 0 \leq y_2 \leq 1, \ 0 \leq y_1 + y_2 \leq 1 \} \) is the two-dimensional support set of \((Y_1, Y_2)\). Sketch a picture of \( R \).

(b) Describe what \( f_{Y_1, Y_2}(y_1, y_2) \) looks like geometrically.

(c) Compute \( \text{Cov}(Y_1, Y_2) \).

(d) Are \( Y_1 \) and \( Y_2 \) independent?