GROUND RULES:

- Print your name at the top of this page.
- This is a closed-book and closed-notes exam.
- You may use a calculator. **Translation:** Show all of your work; use a calculator only to do final calculations and/or to check your work.
- This exam contains 15 questions. Each question is worth 8 points. This exam is worth 120 points.
- Some questions may contain subparts. On each question, there is opportunity for partial credit, so show all of your work and explain all of your reasoning. **Translation:** No work/no explanation means no credit.
- Any discussion or inappropriate communication between you and another examinee, as well as the appearance of any unnecessary material, will result in a very bad outcome for you.
- You have 3 hours to complete this exam.

HONOR PLEDGE FOR THIS EXAM:

After you have finished the exam, please read the following statement and sign your name below it.

> *I promise that I did not discuss any aspect of this exam with anyone other than the instructor, that I neither gave nor received any unauthorized assistance on this exam, and that the work presented herein is entirely my own.*

1. To play the Mega Millions lottery, a player picks 6 numbers from two types of balls:

   - 5 numbers from white balls numbered 1 through 70
   - 1 number from yellow balls numbered 1 through 25 (i.e., the "Mega-ball").

Both types of balls (white and yellow) are drawn at random. White balls are drawn without replacement, and the order of selection does not matter.

The Mega Millions lottery web site says the chance of matching all 6 numbers is "1 in 302,575,350." Use counting techniques to explain where this figure comes from.

2. Suppose the random variable Y has an exponential distribution with mean $\beta > 0$; i.e., the probability density function of Y is

$$f_Y(y) = \begin{cases} \dfrac{1}{\beta} e^{-y/\beta}, & y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Show the median of Y is

$$\phi_{0.5} = \beta \ln 2.$$

3. A random variable Y has moment generating function

$$m_Y(t) = \frac{1}{1-t^2},$$

for $-1 < t < 1$. Find the mean and variance of Y.

4. Suppose Y_1 and Y_2 are continuous random variables with joint probability density function

$$f_{Y_1,Y_2}(y_1,y_2) = \begin{cases} 24y_1y_2, & \begin{array}{c} y_1 > 0,\ y_2 > 0, \\ y_1 + y_2 < 1 \end{array} \\ 0, & \text{otherwise.} \end{cases}$$

Calculate $P(Y_1 < 0.5)$.

5. The number of people arriving for treatment at an emergency room is modeled using a Poisson distribution with mean $\lambda = 5$ per hour.

(a) Calculate the probability at most 2 people arrive for treatment in a particular hour.

(b) Let T denote the time until the second person arrives for treatment. Find the standard deviation of T.

6. Suppose S is a sample space. The events $A_1, A_2, ..., A_n$ are mutually independent with

$$P(A_1) = P(A_2) = \cdots = P(A_n) = p.$$

Derive an expression for the probability exactly y events occur, given that at least one event occurs.

7. A continuous random variable Y has the following probability density function:

$$f_Y(y) = \begin{cases} \dfrac{y}{6}(1+y^2), & 0 < y < 2 \\ 0, & \text{otherwise.} \end{cases}$$

Determine $F_Y(y)$, the cumulative distribution function of Y. This function must be defined for all $y \in \mathbb{R}$.

8. An insurance company offers two types of policies with deductibles Y_1 and Y_2, respectively. The joint probability mass function of Y_1 and Y_2, shown in the table below:

	$y_2 = 1$	$y_2 = 2$	$y_2 = 3$
$y_1 = 1$	0.10	0.10	0.05
$y_1 = 2$	0.15	0.20	0.10
$y_1 = 3$	0.05	0.10	0.15

Find the covariance of Y_1 and Y_2.

9. Suppose Y is a random variable (discrete or continuous) with mean $E(Y) = \mu$ and variance $V(Y) = \sigma^2 > 0$. Suppose $a \neq 0$ is a constant. Find the value of a that minimizes

$$Q(a) = E\left[\left(aY - \frac{1}{a}\right)^2\right].$$

Your solution should depend on μ and σ^2. Verify your solution minimizes $Q(a)$.

10. There are 9 candidates at a debate, including 4 Republicans, 3 Democrats, and 2 Independents. Consider the experiment of lining up the 9 candidates at random.

If each candidate line up is equally likely, what is the probability the candidates within each party are grouped together?

11. During "normal weekday traffic," the time it takes me to drive to work (in minutes) is a continuous random variable Y with cumulative distribution function

$$F_Y(y) = \begin{cases} 0, & y < 30 \\ 1 - e^{-(y-30)/5}, & y \geq 30. \end{cases}$$

Find the expected value of Y.

12. Suppose Y_1 and Y_2 are continuous random variables with joint probability density function

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} \dfrac{16y_2}{y_1^3}, & y_1 > 2,\ 0 < y_2 < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Prove that Y_1 and Y_2 are independent.

13. A study of residents in Fairbanks, Alaska showed the following:

- 20% of the residents smoked
- 0.6% of the residents died of lung cancer
- the probability of death due to lung cancer, given that a resident smoked, was 10 times larger than the probability of death due to lung cancer, given that a resident did not smoke.

A lung cancer death has just been observed in this city. What is the probability the resident who died smoked?

14. The intensity of a hurricane is a random variable Y_1 that is uniformly distributed between 0 and 5.

The damage from a hurricane is a random variable Y_2. Given a hurricane of intensity $Y_1 = y_1$, the random variable Y_2 is distributed as exponential with mean $2y_1$.

Find $V(Y_2)$, the variance of the damage from a random hurricane.

15. A discrete random variable Y has the following probability mass function

$$p_Y(y) = \begin{cases} \dfrac{1}{y \ln 2} \left(\dfrac{1}{2}\right)^y, & y = 1, 2, 3, ... \\ 0, & \text{otherwise.} \end{cases}$$

Find the first moment of Y.