2.6. The random experiment consists of rolling two fair dice. Here is the sample space for this experiment:

$$\begin{aligned} S &= \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(2,2),(2,3),(2,4),(2,5),(2,6), \\ &\quad (3,1),(3,2),(3,3),(3,4),(3,5),(3,6),(4,1),(4,2),(4,3),(4,4),(4,5),(4,6), \\ &\quad (5,1),(5,2),(5,3),(5,4),(5,5),(5,6),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}. \end{aligned}$$

(a) Here are the events listed in the problem:

$$\begin{aligned} A &= \{\text{second die even}\} \\ &= \{(1,2),(1,4),(1,6),(2,2),(2,4),(2,6), \\ &\quad (3,2),(3,4),(3,6),(4,2),(4,4),(4,6), \\ &\quad (5,2),(5,4),(5,6),(6,2),(6,4),(6,6)\} \end{aligned}$$

$$\begin{aligned} B &= \{\text{sum is even}\} \\ &= \{(1,1),(1,3),(1,5),(2,2),(2,4),(2,6), \\ &\quad (3,1),(3,3),(3,5),(4,2),(4,4),(4,6), \\ &\quad (5,1),(5,3),(5,5),(6,2),(6,4),(6,6)\} \end{aligned}$$

$$\begin{aligned} C &= \{\text{at least one number is odd}\} \\ &= \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(2,3),(2,5), \\ &\quad (3,1),(3,2),(3,3),(3,4),(3,5),(3,6),(4,1),(4,3),(4,5), \\ &\quad (5,1),(5,2),(5,3),(5,4),(5,5),(5,6),(6,1),(6,3),(6,5)\}. \end{aligned}$$

(b) The sample points in the event A are listed above. Also,

$$\begin{aligned} \overline{C} &= \{\text{both numbers are even}\} \\ &= \{(2,2),(2,4),(2,6),(4,2),(4,4),(4,6),(6,2),(6,4),(6,6)\}. \end{aligned}$$

$$\begin{aligned} A \cap B &= \{\text{second die even } \textbf{and} \text{ sum is even}\} \\ &= \{(2,2),(2,4),(2,6),(4,2),(4,4),(4,6),(6,2),(6,4),(6,6)\}. \end{aligned}$$

Note that $A \cap B = \overline{C}$. Also,

$$\begin{aligned} A \cap \overline{B} &= \{\text{second die even } \textbf{and} \text{ sum is odd}\} \\ &= \{(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\}. \end{aligned}$$

$$\begin{aligned} \overline{A} \cup B &= \{\text{second die odd } \textbf{or} \text{ sum is even}\} \\ &= \{(1,1),(1,3),(1,5),(2,1),(2,2),(2,3),(2,4),(2,5),(2,6), \\ &\quad (3,1),(3,3),(3,5),(4,1),(4,2),(4,3),(4,4),(4,5),(4,6), \\ &\quad (5,1),(5,3),(5,5),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}. \end{aligned}$$

$$\begin{aligned} \overline{A} \cap C &= \{\text{second die odd } \textbf{and} \text{ at least one number odd}\} \\ &= \{(1,1),(1,3),(1,5),(2,1),(2,3),(2,5), \\ &\quad (3,1),(3,3),(3,5),(4,1),(4,3),(4,5), \\ &\quad (5,1),(5,3),(5,5),(6,1),(6,3),(6,5)\}. \end{aligned}$$

Note that $\overline{A} \subset C$, so $\overline{A} \cap C = \overline{A}$.

2.11. (a) We know that $P(E_1) + P(E_2) + P(E_3) + P(E_4) + P(E_5) = 1$. Therefore,

$$0.15 + 0.15 + 0.4 + P(E_4) + 0.5P(E_4) = 1 \implies 1.5P(E_4) = 0.3.$$

Therefore, $P(E_4) = 0.2$.
(b) We are given that each simple event E_3, E_4, E_5 has the same probability. We are given $P(E_1) = 0.3$ and $P(E_2) = 0.1$. Therefore, $P(E_3)$, $P(E_4)$, and $P(E_5)$ must each be 0.2.

2.17. Define the events

$$\begin{aligned} A &= \{\text{defective shaft}\} \\ B &= \{\text{defective bushing}\}. \end{aligned}$$

We are given $P(A) = 0.08$, $P(B) = 0.06$, and $P(A \cap B) = 0.02$.

(a) $P(B) = 0.06$
(b) We want $P(A \cup B)$. Use additive rule:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.08 + 0.06 - 0.02 = 0.12.$$

(c) "Exactly one" corresponds to the symmetric difference $A \triangle B = (A \cap \overline{B}) \cup (\overline{A} \cap B)$; see Exercise 2.131. First note that $A \cap \overline{B}$ and $\overline{A} \cap B$ are mutually exclusive events. Therefore, the probability of exactly one defect is

$$P(A \triangle B) = P(A \cap \overline{B}) + P(\overline{A} \cap B)$$

by Axiom 3 (i.e., additivity of mutually exclusive events). It is easy to show

$$P(A) = P(A \cap B) + P(A \cap \overline{B}) \implies P(A \cap \overline{B}) = 0.06.$$

Similarly,

$$P(B) = P(A \cap B) + P(\overline{A} \cap B) \implies P(\overline{A} \cap B) = 0.04.$$

Therefore, the probability of exactly one defect is $P(A \triangle B) = 0.1$.
(d) "Neither type of defect" corresponds to $\overline{A} \cap \overline{B} = \overline{A \cup B}$, by DeMorgan's Law. We have

$$P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B) = 0.88.$$

Remark: Two students pointed out that the wording in the problem suggested the events A and B should be interpreted as follows:

$$\begin{aligned} A &= \{\text{defective shaft only}\} \\ B &= \{\text{defective bushing only}\} \end{aligned}$$

(and I agree with them). The difference is that A occurs when an assembly has *only* a shaft defect and B occurs when assembly has *only* a bushing defect. This changes $P(A)$ to $P(A) = 0.10$ and $P(B)$ to $P(B) = 0.08$. The solutions above would be modified to incorporate this change.

2.19. (a) Envision the ordering of supplies from vendors V_1, V_2, or V_3 on two successive days as a random experiment. Here is the sample space:

$$S = \{(V_1, V_1), (V_1, V_2), (V_1, V_3), (V_2, V_1), (V_2, V_2), (V_2, V_3), (V_3, V_1), (V_3, V_2), (V_3, V_3)\}.$$

There are $N = 9$ outcomes (sample points) in S.
(b) If vendors are selected at random, then each outcome in S is equally likely. Therefore, the probability of each outcome is 1/9.
(c) Define the events

$$\begin{aligned} A &= \{\text{same vendor on both days}\} \\ B &= \{\text{vendor 2 gets at least one order}\}. \end{aligned}$$

Here are the outcomes these events:

$$\begin{aligned} A &= \{(V_1, V_1), (V_2, V_2), (V_3, V_3)\} \\ B &= \{(V_1, V_2), (V_2, V_1), (V_2, V_2), (V_2, V_3), (V_3, V_2)\}. \end{aligned}$$

Also,

$$\begin{aligned} A \cup B &= \{V_1, V_1), (V_1, V_2), (V_2, V_1), (V_2, V_2), (V_2, V_3), (V_3, V_2), (V_3, V_3)\} \\ A \cap B &= \{(V_2, V_2)\}. \end{aligned}$$

Because each outcome is equally likely,

$$P(A) = \frac{3}{9}, \quad P(B) = \frac{5}{9}, \quad P(A \cup B) = \frac{7}{9}, \text{ and } P(A \cap B) = \frac{1}{9}.$$

2.27. (a) Let r, l, and s denote right, left, and straight turns, respectively. The sample space for two cars' movements is

$$S = \{(r, r), (r, l), (r, s), (l, r), (l, l), (l, s), (s, r), (s, l), (s, s)\},$$

where, for example, (r, r) means car 1 turns right and car 2 turns right. There are $N = 9$ outcomes in S.
(b) Define the event

$$A = \{\text{at least one car turns left}\} = \{(r, l), (l, r), (l, l), (l, s), (s, l)\}.$$

There are $n_a = 5$ outcomes (sample points) in A. Assuming each outcome is equally likely,

$$P(A) = \frac{5}{9}.$$

(c) Define the event

$$B = \{\text{at most one car turns}\} = \{(r, s), (l, s), (s, r), (s, l), (s, s)\}.$$

There are $n_b = 5$ outcomes in B. Assuming each outcome is equally likely,

$$P(B) = \frac{5}{9}.$$

2.30. (a) We can think of an outcome (sample point) as (w_1, w_2, w_3), where w_1 is the ranking for wine 1, w_2 is the ranking for wine 2, and w_3 is the ranking for wine, respectively. For example, the outcome $(2, 1, 3)$ means wine 1 was ranked 2nd, wine 2 was ranked 1st, and wine 3 was ranked 3rd.
(b) The sample space from the wine expert's judging is

$$S = \{(1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1)\}.$$

There are $N = 6$ outcomes in S.
(c) If the "expert" really is no expert at all (and rankings are assigned at random), this suggests that each outcome in S is equally likely. Without loss, suppose wine 1 truly is the best wine and wine 2 truly is the 2nd best wine. Define the event

$$A = \{\text{"expert" ranks wine 1 ahead of wine 2}\} = \{(1,2,3),(1,3,2),(2,3,1)\}.$$

There are $n_a = 3$ outcomes in A. Assuming each outcome is equally likely,

$$P(A) = \frac{3}{6} = 0.5.$$

This would be expected if the "expert" knows nothing about wine.

Remark: A student pointed out that she interpreted part (c) differently than I did; simply that the best wine is rated in the top 2 choices. In retrospect, I think I agree with her, and this will change the answers slightly.

2.89. (a) The smallest $P(A \cap B)$ can be is 0. This happens when A and B are mutually exclusive.
(b) Note that $A \cap B \subseteq A$. Therefore, $P(A \cap B) \leq P(A)$ by the monotonicity rule. Similarly, $A \cap B \subseteq B$ so $P(A \cap B) \leq P(B)$. Therefore,

$$P(A \cap B) \leq \min\{P(A), P(B)\}.$$

Note: The restriction "$P(A) + P(B) < 1$" is needed in part (a). If $P(A \cap B) = 0$, then $P(A \cup B) = P(A) + P(B)$, and we know this cannot exceed 1.

2.106. Use Bonferroni's Inequality:

$$P(A \cap B) \geq 1 - P(\overline{A}) - P(\overline{B}) = P(A) + P(B) - 1.$$

If A and B are equally likely events, then $P(A) = P(B)$. Therefore,

$$P(A \cap B) \geq 2P(A) - 1.$$

Using $P(A) = 0.99$ gives

$$P(A \cap B) \geq 2(0.99) - 1 = 0.98.$$

2.131. The symmetric difference

$$A \triangle B = (A \cap \overline{B}) \cup (\overline{A} \cap B).$$

In words, this means that "exactly 1 of A and B occurs" (but both cannot occur). Clearly, $A \cap \overline{B}$ and $\overline{A} \cap B$ are mutually exclusive. Therefore,

$$P(A \triangle B) = P(A \cap \overline{B}) + P(\overline{A} \cap B)$$

by Axiom 3. Now, note that

$$A = \underbrace{(A \cap \overline{B}) \cup (A \cap B)}_{\text{mutually exclusive}} \implies P(A) = P(A \cap \overline{B}) + P(A \cap B).$$

Similarly,

$$B = \underbrace{(\overline{A} \cap B) \cup (A \cap B)}_{\text{mutually exclusive}} \implies P(B) = P(\overline{A} \cap B) + P(A \cap B).$$

Therefore,

$$\begin{aligned} P(A \triangle B) &= P(A \cap \overline{B}) + P(\overline{A} \cap B) \\ &= [P(A) - P(A \cap B)] + [P(B) - P(A \cap B)] \\ &= P(A) + P(B) - 2P(A \cap B). \end{aligned}$$

2.144. (a,b) The sample space S contains 4 sample points, denoted by E_1, E_2, E_3, and E_4; i.e.,

$$S = \{E_1, E_2, E_3, E_4\}.$$

There are $2^4 = 16$ possible events:

$$\emptyset, \{E_1\}, \{E_2\}, \{E_3\}, \{E_4\}, \{E_1, E_2\}, \{E_1, E_3\}, \{E_1, E_4\}, \{E_2, E_3\}, \{E_2, E_4\}, \{E_3, E_4\}$$
$$\{E_1, E_2, E_3\}, \{E_1, E_2, E_4\}, \{E_1, E_3, E_4\}, \{E_2, E_3, E_4\}, S$$

Note that there are

$$\begin{aligned} \binom{4}{0} &= \text{1 event with no sample points (the null set)} \\ \binom{4}{1} &= \text{4 events with 1 sample point} \\ \binom{4}{2} &= \text{6 events with 2 sample points} \\ \binom{4}{3} &= \text{4 events with 3 sample points} \\ \binom{4}{4} &= \text{1 event with 4 sample points (the entire sample space).} \end{aligned}$$

Note that $2^4 = \binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4}$.
(c) Define

$$\begin{aligned} A &= \{E_1, E_2, E_3\} \\ B &= \{E_2, E_4\}. \end{aligned}$$

Then,

$$\begin{aligned} A \cup B &= \{E_1, E_2, E_3, E_4\} = S \\ A \cap B &= \{E_2\}. \end{aligned}$$

By DeMorgan's Law,

$$\overline{A} \cap \overline{B} = \overline{A \cup B} = \overline{S} = \emptyset.$$

Finally,

$$\overline{A} \cup B = \{E_4\} \cup \{E_2, E_4\} = \{E_2, E_4\}.$$