2.37. (a) Think of each city as an "object." Each one is distinct. Therefore, there are $6! = 720$ different itineraries.
(b) Envision the process of selecting an itinerary as a random experiment with sample space S. There are $N = 720$ outcomes (sample points) in S and each one has the following form:

$$(\_\_ \; \_\_ \; \_\_ \; \_\_ \; \_\_ \; \_\_).$$

Because the businesswoman "randomly selects," we can assume each outcome in S is equally likely. Define

$$A = \{\text{visit Denver before SF}\}.$$

Denote the other 4 cities as O_1, O_2, O_3, and O_4.

<u>Case 1</u>: If Denver is the first city visited, then the outcome looks like:

$$(\underline{\text{ D }} \; \_\_ \; \_\_ \; \_\_ \; \_\_ \; \_\_).$$

There are $5! = 120$ outcomes in A that look like this; just permute the other 5 cities.

<u>Case 2</u>: If Denver is the second city visited, then the outcome looks like:

$$(\_\_ \; \underline{\text{ D }} \; \_\_ \; \_\_ \; \_\_ \; \_\_).$$

How many outcomes in A look like this?

$$\begin{aligned} n_1 &= \text{select later position for SF} = 4 \\ n_2 &= \text{permute the other 4 cities} = 4! \end{aligned}$$

By the basic rule of counting, there are $4 \times 4! = 96$ outcomes in A that look like this.

<u>Case 3</u>: If Denver is the third city visited, then the outcome looks like:

$$(\_\_ \; \_\_ \; \underline{\text{ D }} \; \_\_ \; \_\_ \; \_\_).$$

How many outcomes in A look like this?

$$\begin{aligned} n_1 &= \text{select later position for SF} = 3 \\ n_2 &= \text{permute the other 4 cities} = 4! \end{aligned}$$

By the basic rule of counting, there are $3 \times 4! = 72$ outcomes in A that look like this.

<u>Case 4</u>: If Denver is the fourth city visited, then the outcome looks like:

$$(\_\_ \; \_\_ \; \_\_ \; \underline{\text{ D }} \; \_\_ \; \_\_).$$

How many outcomes in A look like this?

$$\begin{aligned} n_1 &= \text{select later position for SF} = 2 \\ n_2 &= \text{permute the other 4 cities} = 4! \end{aligned}$$

By the basic rule of counting, there are $2 \times 4! = 48$ outcomes in A that look like this.

Case 5: If Denver is the fifty city visited, then the outcome looks like:

$$(\_\_ \; \_\_ \; \_\_ \; \_\_ \; \underline{\text{D}} \; \_\_).$$

How many outcomes in A look like this?

$$\begin{aligned} n_1 &= \text{select later position for SF} = 1 \\ n_2 &= \text{permute the other 4 cities} = 4! \end{aligned}$$

By the basic rule of counting, there are $1 \times 4! = 24$ outcomes in A that look like this.

Add up the outcomes in A by examining all 5 cases. There are

$$n_a = 120 + 96 + 72 + 48 + 24 = 360.$$

Therefore, assuming each itinerary is equally likely,

$$P(A) = \frac{n_a}{N} = \frac{360}{720} = 0.5.$$

2.43. Here, we have 9 distinct objects (taxis), and we want to partition them three distinct groups containing $n_1 = 3$, $n_2 = 5$, and $n_3 = 1$ objects. The first group goes to airport A. The second group goes to airport B. The third group goes to airport C. There are

$$\binom{9}{3\ 5\ 1} = \frac{9!}{3!\ 5!\ 1!} = 504$$

ways this can be accomplished.

2.44. (a) We now assume taxis are dispatched to airports at random. This means that each taxi has a 1/3 chance of being dispatched to airport A, 1/3 to airport B, and 1/3 to airport C. The probability a taxi "in need of repair" is dispatched to airport C is 1/3.
(b) All we have to think about is the three taxis that are "in need of repair." For example, the destinations for these three taxis could be all to airport A; i.e.,

$$(\underline{\text{ A }} \; \underline{\text{ A }} \; \underline{\text{ A }})$$

There are $3^3 = 27$ ways these three taxis could be dispatched to the three airports (i.e., there are 3 possibilities for each position). Now, for each airport to receive a taxi, we need destinations like

$$(\underline{\text{ B }} \; \underline{\text{ A }} \; \underline{\text{ C }})$$

There are $3! = 6$ ways to permute these three distinct objects (i.e., ABC, ACB, BAC, BCA, CAB, CBA). The answer is therefore 6/27.

2.52. This is an example of the basic rule of counting. There are

$$\begin{aligned} n_1 &= 3 && \text{settings for temperature} \\ n_2 &= 3 && \text{settings for pressure} \\ n_3 &= 2 && \text{types of catalysts.} \end{aligned}$$

Therefore, there are $3 \times 3 \times 2 = 18$ different runs needed to capture all possible combinations of these 3 factors. In experimental design (in statistics), these 18 runs would constitute 1 replication of a $3 \times 3 \times 2$ factorial experiment.

2.55. (a) Order is not important here in the selection. Therefore, the number of ways to choose 10 nurses from 90 is simply

$$\binom{90}{10} = \frac{90!}{10!\ 80!} = 5{,}720{,}645{,}481{,}903.$$

In R,

```
> choose(90,10)
[1] 5.720645e+12
```

(b) Define the event

$$A = \{\text{exactly 4 male nurses (among 10 selected)}\}.$$

From the 20 male nurses, we choose 4. From the 70 female nurses, we choose 6. By the basic rule of counting, there are

$$n_a = \binom{20}{4}\binom{70}{6} = 635{,}256{,}947{,}325$$

ways to do this. In R,

```
> choose(20,4)*choose(70,6)
[1] 635256947325
```

Assuming each selection is equally likely, the probability of getting exactly 4 male nurses is

$$P(A) = \frac{n_a}{N} = \frac{635{,}256{,}947{,}325}{5{,}720{,}645{,}481{,}903} \approx 0.111.$$

2.63. There are 8 members of the HR Advisory Board. The number of ways 5 votes "in favor" of the plaintiff can result among the 8 members is

$$\binom{8}{5} = \frac{8!}{5!\ 3!} = 56$$

i.e., you are just choosing 5 members to vote in favor (from the 8 members). A potential sample point looks like

$$(\ \underline{\text{M}}\ \ \underline{\text{W}}\ \ \underline{\text{M}}\ \ \underline{\text{W}}\ \ \underline{\text{W}}\ ).$$

How many sample points have all 5 women? Just the one that looks like this:

$$(\ \underline{\text{W}}\ \ \underline{\text{W}}\ \ \underline{\text{W}}\ \ \underline{\text{W}}\ \ \underline{\text{W}}\ ).$$

Therefore, if each of the 56 outcomes is equally likely—which could correspond to there being no sex bias—then the probability the 5-3 vote splits down gender lines is 1/56. Obviously, this is a small number, which would suggest perhaps there was bias. However, this event is certainly possible even when there is no bias.

2.64. We are considering outcomes (sample points) that look like this:

$$(\_\_ \; \_\_ \; \_\_ \; \_\_ \; \_\_ \; \_\_).$$

Each position holds a die outcome. There are 6 outcomes for the 1st die, 6 for the 2nd die, and so on. By the basic rule of counting, there are

$$N = 6^6 = 46656$$

outcomes from tossing a die 6 times. Now, define the event

$$A = \{1,\ 2,\ 3,\ 4,\ 5,\ 6 \text{ appearing in any order}\}.$$

How many outcomes are in A? We are considering sample points (for A) that look like this:

$$(\underline{\ 4\ } \; \underline{\ 3\ } \; \underline{\ 6\ } \; \underline{\ 5\ } \; \underline{\ 1\ } \; \underline{\ 2\ }).$$

There are $n_a = 6! = 720$ outcomes in A; i.e., this is the number of ways to permute 6 "distinct objects." Assuming each outcome is equally likely,

$$P(A) = \frac{n_a}{N} = \frac{6!}{6^6} = \frac{720}{46656} \approx 0.0154.$$

2.68. (a)

$$\binom{n}{n} = \frac{n!}{n!(n-n)!} = \frac{n!}{n!\ 0!} \overset{0!=1}{=} \frac{n!}{n!} = 1$$

Interpretation: There is 1 way to choose n objects from n (i.e., you select all of them, and there is only 1 way to do this).

(b)

$$\binom{n}{0} = \frac{n!}{0!(n-0)!} = \frac{n!}{0!\ n!} \overset{0!=1}{=} \frac{n!}{n!} = 1$$

Interpretation: There is 1 way to choose 0 objects from n (i.e., you do not select any, and there is only 1 way to do this).

(c)

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n!}{(n-r)!\ r!} = \binom{n}{n-r}$$

Interpretation: The number of ways to choose r objects from n is the same as the number of ways to select $n-r$ objects from n. This makes sense; by selecting r "chosen" objects, you are at the same time implicitly selecting those objects 'not chosen."

(d) Recall the binomial expansion of $(a+b)^n$; i.e.,

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i.$$

This holds for all $a, b \in \mathbb{R}$, so take $a = b = 1$. The LHS $= 2^n$. The RHS equals $\sum_{i=0}^{n} \binom{n}{i}$.

2.162. This is similar to the flag problem in the notes. We can think of one outcome (sample point) in the underlying sample space S as having the following structure:

$$(\_\_ \; \_\_ \; \_\_ \; \_\_ \; \_\_ \; \_\_ \; \_\_ \; \_\_ \; \_\_)$$

Treat the objects (i.e., cars) as only distinguishable by car type (S, D, or C), that is, one sample point looks like this:

$$(\underline{\text{ S }} \; \underline{\text{ D }} \; \underline{\text{ D }} \; \underline{\text{ C }} \; \underline{\text{ S }} \; \underline{\text{ S }} \; \underline{\text{ C }} \; \underline{\text{ D }} \; \underline{\text{ D }}).$$

Define

$$A = \{\text{sports cars (S) grouped together}\}.$$

and assume that outcomes (sample points) are equally likely. The number of sample points in S is

$$N = \binom{9}{3\;3\;3} = \frac{9!}{3!\;3!\;3!} = 1680.$$

This is the number of ways to permute 9 objects, of which 3 are "alike," 3 are "alike," and 3 are "alike." Now, we need to count the number of sample points in A. We can do this using the basic counting rule:

$$\begin{aligned} n_1 &= \text{number of ways to select 3 adjacent positions for sports cars (S)} = 7 \\ n_2 &= \text{number of ways to permute domestic/compact cars among remaining positions} = \binom{6}{3\;3} \end{aligned}$$

Therefore,

$$n_a = n_1 \times n_2 = 7 \times \binom{6}{3\;3} = 140$$

and

$$P(A) = \frac{n_a}{N} = \frac{140}{1680} \approx 0.0833.$$

2.169. (a) It's pretty easy to think of what an outcome (sample point) would look like:

$$(\underbrace{\_\_ \; \_\_ \; \_\_ \; \_\_}_{\text{Drinker I}} \quad \underbrace{\_\_ \; \_\_ \; \_\_ \; \_\_}_{\text{Drinker II}} \quad \underbrace{\_\_ \; \_\_ \; \_\_ \; \_\_}_{\text{Drinker III}})$$

Each drinker ranks 4 beers (A, B, C, and D). For example, one outcome looks like this:

$$(\underbrace{\underline{1} \; \underline{2} \; \underline{3} \; \underline{4}}_{\text{Drinker I}} \quad \underbrace{\underline{4} \; \underline{3} \; \underline{2} \; \underline{1}}_{\text{Drinker II}} \quad \underbrace{\underline{2} \; \underline{1} \; \underline{4} \; \underline{3}}_{\text{Drinker III}}).$$

This would mean Drinker I rated A first, B second, C third, and D fourth. Similar interpretations hold for the other drinkers. There are

$$N = 4! \times 4! \times 4! = 13824$$

outcomes in S.
(b) We are told to assume each outcome in S is equally likely. Define

$$A = \{\text{a beer receives a combined (summed) ranking of 4 or less}\}.$$

Our job now is to count how many ways the event A can occur. We can use the basic counting rule to do this:

$$\begin{aligned}
n_1 &= \text{choose 1 beer} = 4 \\
n_2 &= \text{number of ways this beer can get a summed rating of 4 or less} = 4 \\
n_3 &= \text{number of ways to permute the three beers not getting the high rating} = (3!)^3 = 216
\end{aligned}$$

Clearly, $n_1 = 4$; there are only 4 beers, and we choose 1 of them. For n_2, there are 4 ways that a given beer can be given a rating of 4 or less:

$$\begin{array}{lll}
\text{Drinker I, Drinker II, Drinker III:} & 1,1,1 & \text{(summed rating} = 3) \\
\text{Drinker I, Drinker II, Drinker III:} & 1,1,2 & \text{(summed rating} = 4) \\
\text{Drinker I, Drinker II, Drinker III:} & 1,2,1 & \text{(summed rating} = 4) \\
\text{Drinker I, Drinker II, Drinker III:} & 2,1,1 & \text{(summed rating} = 4)
\end{array}$$

For n_3, suppose each drinker rates beer A the best; i.e., we have a sample point with this structure:

$$(\underbrace{\underline{1}\ \underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }}_{\text{Drinker I}}\quad \underbrace{\underline{1}\ \underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }}_{\text{Drinker II}}\quad \underbrace{\underline{1}\ \underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }}_{\text{Drinker III}}).$$

For drinker I, there are $3! = 6$ different ways he can order the "other beers;" i.e., (234, 243, 324, 342, 423, and 432) and similarly for drinkers 2 and 3. Therefore,

$$n_a = 4 \times 4 \times 216 = 3456$$

and

$$P(A) = \frac{n_a}{N} = \frac{3456}{13824} = 0.25.$$