3.6. Let's write out the sample space for this random experiment:

$$S = \{(1,2),(1,3),(1,4),(1,5),(2,3),(2,4),(2,5),(3,4),(3,5),(4,5)\}.$$

This sample space assumes the ordering of the balls selected does not matter. There are

$$N = \binom{5}{2} = 10$$

outcomes (sample points) in S. In this problem, we assume the outcomes are equally likely.

(a) Define

$$Y = \text{ largest of the two sampled numbers.}$$

We can calculate the value of Y for each outcome $\omega \in S$:

Outcome $\omega \in S$	$Y(\omega) = y$	Probability $P(\{\omega\})$
$(1,2)$	$Y((1,2)) = 2$	$\frac{1}{10}$
$(1,3)$	$Y((1,3)) = 3$	$\frac{1}{10}$
$(1,4)$	$Y((1,4)) = 4$	$\frac{1}{10}$
$(1,5)$	$Y((1,5)) = 5$	$\frac{1}{10}$
$(2,3)$	$Y((2,3)) = 3$	$\frac{1}{10}$
$(2,4)$	$Y((2,4)) = 4$	$\frac{1}{10}$
$(2,5)$	$Y((2,5)) = 5$	$\frac{1}{10}$
$(3,4)$	$Y((3,4)) = 4$	$\frac{1}{10}$
$(3,5)$	$Y((3,5)) = 5$	$\frac{1}{10}$
$(4,5)$	$Y((4,5)) = 5$	$\frac{1}{10}$

The pmf of Y is given by

$$p_Y(y) = P(Y = y) = P(\{\text{all } \omega \in S : Y(\omega) = y\}) = \sum_{\substack{\omega \in S \\ Y(\omega)=y}} P(\{\omega\}),$$

which can be depicted in the following table:

y	2	3	4	5
$p_Y(y)$	1/10	2/10	3/10	4/10

It is easy to see that this is a valid pmf.

(b) Define

$$Y = \text{ sum of the two sampled numbers.}$$

We can calculate the value of Y for each outcome $\omega \in S$ just like we did in part (a); see next page.

Outcome $\omega \in S$	$Y(\omega) = y$	Probability $P(\{\omega\})$
$(1,2)$	$Y((1,2)) = 3$	$\frac{1}{10}$
$(1,3)$	$Y((1,3)) = 4$	$\frac{1}{10}$
$(1,4)$	$Y((1,4)) = 5$	$\frac{1}{10}$
$(1,5)$	$Y((1,5)) = 6$	$\frac{1}{10}$
$(2,3)$	$Y((2,3)) = 5$	$\frac{1}{10}$
$(2,4)$	$Y((2,4)) = 6$	$\frac{1}{10}$
$(2,5)$	$Y((2,5)) = 7$	$\frac{1}{10}$
$(3,4)$	$Y((3,4)) = 7$	$\frac{1}{10}$
$(3,5)$	$Y((3,5)) = 8$	$\frac{1}{10}$
$(4,5)$	$Y((4,5)) = 9$	$\frac{1}{10}$

The pmf of Y is given by

$$p_Y(y) = P(Y = y) = P(\{\text{all } \omega \in S : Y(\omega) = y\}) = \sum_{\substack{\omega \in S \\ Y(\omega)=y}} P(\{\omega\}),$$

which can be depicted in the following table:

y	3	4	5	6	7	8	9
$p_Y(y)$	1/10	1/10	2/10	2/10	2/10	1/10	1/10

It is easy to see that this is a valid pmf.

3.10. Define the random variable

$$Y = \text{ number of days between a pair of rentals.}$$

Clearly, Y is a nonnegative random variable with support $R = \{y : y = 1, 2, 3, 4, ...\}$. The "one day in five" phrase means the probability of a rental on any day is 0.2 (and hence the probability of there being no rental on any day is 0.8). Independence means we multiply the "each day probabilities" together.

Here are the probabilities associated with each value of y:

y	$p_Y(y) = P(Y = y)$
1	0.2
2	(0.8)0.2
3	(0.8)(0.8)0.2
4	(0.8)(0.8)(0.8)0.2
5	(0.8)(0.8)(0.8)(0.8)0.2
$\vdots$	$\vdots$

A general formula for the pmf of Y is

$$p_Y(y) = P(Y = y) = (0.8)^{y-1}(0.2),$$

for $y = 1, 2, 3, ...$ This is an example of a **geometric** probability distribution. It is easy to check this is a valid pmf.

3.12. The values of $E(Y)$, $E(1/Y)$, and $E(Y^2-1)$ are

$$\begin{aligned} E(Y) &= \sum_{y=1}^{4} y p_Y(y) = 1(0.4)+2(0.3)+3(0.2)+4(0.1) = 2 \\ E\left(\frac{1}{Y}\right) &= \sum_{y=1}^{4} \frac{1}{y} p_Y(y) = \frac{1}{1}(0.4)+\frac{1}{2}(0.3)+\frac{1}{3}(0.2)+\frac{1}{4}(0.1) \approx 0.642 \\ E(Y^2-1) &= \sum_{y=1}^{4} (y^2-1) p_Y(y) \\ &= (1^2-1)(0.4)+(2^2-1)(0.3)+(3^2-1)(0.2)+(4^2-1)(0.1) = 4. \end{aligned}$$

Another way to calculate $E(Y^2-1)$ is to first calculate the second moment:

$$E(Y^2) = \sum_{y=1}^{4} y^2 p_Y(y) = 1^2(0.4)+2^2(0.3)+3^2(0.2)+4^2(0.1) = 5.$$

Then observe that

$$\begin{aligned} E(Y^2-1) &= E(Y^2)-1 \\ &= 5-1 = 4. \end{aligned}$$

Finally, we can get $V(Y)$ by using the variance computing formula

$$V(Y) = E(Y^2) - [E(Y)]^2 = 5 - 2^2 = 1.$$

3.21. First we can calculate $E(R^2)$, where R is the distance (in blocks) the fire engine can cover. The pmf of R, $p_R(r)$, is given in the problem. Therefore,

$$E(R^2) = \sum_{r=21}^{26} r^2 p_R(r) = 21^2(0.05)+22^2(0.20)+23^2(0.30)+24^2(0.25)+25^2(0.15)+26^2(0.05) = 549.1.$$

With $C = 8$, the expected number of residential homes that can be served is

$$E(N) = E(8\pi R^2) = 8\pi E(R^2) = 8\pi(549.1) \approx 13800.39,$$

or about 13,800 homes.

3.29. There is a typo in the problem; it should read

$$E(Y) = \sum_{k=1}^{\infty} P(Y \geq k);$$

i.e., the index of summation should be "k," not "i." Because the support is $R = \{1, 2, 3, ..., \}$, we can write out what the probabilities $P(Y \geq k)$ are; see next page.

$$
\begin{array}{rcr}
P(Y \geq 1) & = & P(Y=1)+P(Y=2)+P(Y=3)+P(Y=4)+P(Y=5)+P(Y=6)+\cdots \\
P(Y \geq 2) & = & P(Y=2)+P(Y=3)+P(Y=4)+P(Y=5)+P(Y=6)+\cdots \\
P(Y \geq 3) & = & P(Y=3)+P(Y=4)+P(Y=5)+P(Y=6)+\cdots \\
P(Y \geq 4) & = & P(Y=4)+P(Y=5)+P(Y=6)+\cdots \\
P(Y \geq 5) & = & P(Y=5)+P(Y=6)+\cdots \\
P(Y \geq 6) & = & P(Y=6)+\cdots \\
\vdots & \vdots & \vdots
\end{array}
$$

and the pattern continues. Now, "add down." There is 1 $P(Y=1)$ term. There are 2 $P(Y=2)$ terms. There are 3 $P(Y=3)$ terms. There are 4 $P(Y=4)$ terms. There are 5 $P(Y=5)$ terms. There are 6 $P(Y=6)$ terms, and so on. Therefore,

$$
\begin{aligned}
\sum_{k=1}^{\infty} P(Y \geq k) &= P(Y \geq 1)+P(Y \geq 2)+P(Y \geq 3)+P(Y \geq 4)+P(Y \geq 5)+P(Y \geq 6)+\cdots \\
&= 1P(Y=1)+2P(Y=2)+3P(Y=3)+4P(Y=4)+5P(Y=5)+6P(Y=6)+\cdots \\
&= \sum_{k=1}^{\infty} kP(Y=k),
\end{aligned}
$$

which is the definition of $E(Y)$ when Y has support $R=\{1,2,3,...,\}$.

3.33. Suppose Y is a discrete random variable with pmf $p_Y(y)$ and support R.

(a) Using the definition of mathematical expectation,

$$
\begin{aligned}
E(aY+b) = \sum_{y \in R}(ay+b)p_Y(y) &= \sum_{y \in R} ayp_Y(y) + \sum_{y \in R} bp_Y(y) \\
&= a\sum_{y \in R} yp_Y(y) + b\sum_{y \in R} p_Y(y).
\end{aligned}
$$

However, $\sum_{y \in R} p_Y(y) = 1$ and $\sum_{y \in R} yp_Y(y) = E(Y)$. This shows

$$E(aY+b) = aE(Y)+b.$$

(b) Define $X = aY+b$. We want to show $V(X) = a^2V(Y)$. From part (a), we know $E(X) = E(aY+b) = aE(Y)+b$. Also,

$$E(X^2) = E[(aY+b)^2] = E(a^2Y^2+2abY+b^2) = a^2E(Y^2)+2abE(Y)+b^2$$

and

$$[E(X)]^2 = [aE(Y)+b]^2 = a^2[E(Y)]^2+2abE(Y)+b^2.$$

Therefore, from the variance computing formula, we have

$$
\begin{aligned}
V(X) = E(X^2)-[E(X)]^2 &= a^2E(Y^2)-a^2[E(Y)]^2 \\
&= a^2\{E(Y^2)-[E(Y)]^2\} = a^2V(Y).
\end{aligned}
$$

Note that in this part, we did not assume that Y was discrete; in fact, this result holds for any random variable with finite variance.

3.154. (a) The first derivative of $m_Y(t) = [(1/3)e^t + (2/3)]^5$ is

$$\frac{d}{dt}\left(\frac{1}{3}e^t + \frac{2}{3}\right)^5 = 5\left(\frac{1}{3}e^t + \frac{2}{3}\right)^4 \times \frac{1}{3}e^t = \frac{5}{3}e^t\left(\frac{1}{3}e^t + \frac{2}{3}\right)^4.$$

Therefore,

$$E(Y) = \frac{d}{dt}m_Y(t)\bigg|_{t=0} = \frac{5}{3}e^0\left(\frac{1}{3}e^0 + \frac{2}{3}\right)^4 = \frac{5}{3}.$$

The second derivative of $m_Y(t) = [(1/3)e^t + (2/3)]^5$ is

$$\frac{d^2}{dt^2}\left(\frac{1}{3}e^t + \frac{2}{3}\right)^5 = \frac{d}{dt}\left[\frac{5}{3}e^t\left(\frac{1}{3}e^t + \frac{2}{3}\right)^4\right] = \frac{5}{3}e^t\left(\frac{1}{3}e^t + \frac{2}{3}\right)^4 + \frac{5}{3}e^t \times 4\left(\frac{1}{3}e^t + \frac{2}{3}\right)^3 \times \frac{5}{3}e^t.$$

Therefore,

$$\begin{aligned} E(Y^2) = \frac{d^2}{dt^2}m_Y(t)\bigg|_{t=0} &= \frac{5}{3}e^0\left(\frac{1}{3}e^0 + \frac{2}{3}\right)^4 + \frac{5}{3}e^0 \times 4\left(\frac{1}{3}e^0 + \frac{2}{3}\right)^3 \times \frac{1}{3}e^0 \\ &= \frac{5}{3} + \frac{20}{9} = \frac{35}{9}. \end{aligned}$$

From the variance computing formula,

$$V(Y) = E(Y^2) - [E(Y)]^2 = \frac{35}{9} - \left(\frac{5}{3}\right)^2 = \frac{10}{9}.$$

(b) We calculated $E(Y)$ and $V(Y)$ with this mgf in the notes; see Example 3.13.
(c) The first derivative of $m_Y(t) = \exp[2(e^t - 1)]$ is

$$\frac{d}{dt}\exp[2(e^t - 1)] = 2e^t\exp[2(e^t - 1)].$$

Therefore,

$$E(Y) = \frac{d}{dt}m_Y(t)\bigg|_{t=0} = 2e^0\exp[2(e^0 - 1)] = 2.$$

The second derivative of $m_Y(t) = \exp[2(e^t - 1)]$ is

$$\frac{d^2}{dt^2}\exp[2(e^t - 1)] = \frac{d}{dt}\,2e^t\exp[2(e^t - 1)] = 2e^t\exp[2(e^t - 1)] + 2e^t \times 2e^t\exp[2(e^t - 1)].$$

Therefore,

$$E(Y^2) = \frac{d^2}{dt^2}m_Y(t)\bigg|_{t=0} = 2e^0\exp[2(e^0 - 1)] + 2e^0 \times 2e^0\exp[2(e^0 - 1)] = 6.$$

From the variance computing formula,

$$V(Y) = E(Y^2) - [E(Y)]^2 = 6 - 2^2 = 2.$$

Note: The mgf of Y in part (a) corresponds to a $b(n = 5, p = 1/3)$ distribution. The mgf of Y in part (b) corresponds to a geometric($p = 1/2$) distribution. The mgf of Y in part (c) corresponds to a Poisson($\lambda = 2$) distribution. You'll see this later.

3.155. You should see that the mgf

$$m_Y(t) = \frac{1}{6}e^t + \frac{2}{6}e^{2t} + \frac{3}{6}e^{3t}$$

corresponds to this pmf:

y	1	2	3
$p_Y(y)$	1/6	2/6	3/6

To see why, just apply the definition of the mgf; i.e., $m_Y(t) = E(e^{tY})$ with this distribution. Therefore, this is your answer to part (c).

To find $E(Y)$ in part (a), we could quickly calculate

$$E(Y) = \sum_{y=1}^{3} y p_Y(y) = 1(1/6) + 2(2/6) + 3(3/6) = \frac{14}{6}.$$

Using the mgf should give us the same answer. The first derivative of $m_Y(t)$ is

$$\frac{d}{dt}\left(\frac{1}{6}e^t + \frac{2}{6}e^{2t} + \frac{3}{6}e^{3t}\right) = \frac{1}{6}e^t + \frac{4}{6}e^{2t} + \frac{9}{6}e^{3t}$$

Therefore,

$$E(Y) = \frac{d}{dt}m_Y(t)\bigg|_{t=0} = \frac{1}{6}e^0 + \frac{4}{6}e^{2(0)} + \frac{9}{6}e^{3(0)} = \frac{14}{6}.$$

To find $V(Y)$ in part (b), we could calculate directly

$$\begin{aligned} V(Y) = \sum_{y=1}^{3}(y-\mu)^2 p_Y(y) &= \sum_{y=1}^{3}\left(y - \frac{14}{6}\right)^2 p_Y(y) \\ &= \left(1 - \frac{14}{6}\right)^2\left(\frac{1}{6}\right) + \left(2 - \frac{14}{6}\right)^2\left(\frac{2}{6}\right) + \left(3 - \frac{14}{6}\right)^2\left(\frac{3}{6}\right) = \frac{5}{9} \end{aligned}$$

using the pmf of Y above. If we wanted to the use the mgf, we could first find $E(Y^2)$. The second derivative of $m_Y(t)$ is

$$\frac{d^2}{dt^2}\left(\frac{1}{6}e^t + \frac{2}{6}e^{2t} + \frac{3}{6}e^{3t}\right) = \frac{d}{dt}\left(\frac{1}{6}e^t + \frac{4}{6}e^{2t} + \frac{9}{6}e^{3t}\right) = \frac{1}{6}e^t + \frac{8}{6}e^{2t} + \frac{27}{6}e^{3t}$$

Therefore,

$$E(Y^2) = \frac{d^2}{dt^2}m_Y(t)\bigg|_{t=0} = \frac{1}{6}e^0 + \frac{8}{6}e^{2(0)} + \frac{27}{6}e^{3(0)} = 6.$$

From the variance computing formula,

$$V(Y) = E(Y^2) - [E(Y)]^2 = 6 - \left(\frac{14}{6}\right)^2 = \frac{5}{9}.$$

3.158. We have a random variable Y with mgf $m_Y(t)$. Now, we consider the linear function

$$W = aY + b.$$

The mgf of W is equal to

$$m_W(t) = E(e^{tW}) = E[e^{t(aY+b)}] = E(e^{atY+bt}) = E(e^{bt}e^{atY}) = e^{bt}E(e^{atY}).$$

The last step is true because e^{bt} is a constant so it factors outside the expectation (all previous steps are just algebra). Now, what is $E(e^{atY})$? Let $s = at$. Then,

$$E(e^{atY}) = E(e^{sY}) = m_Y(s) = m_Y(at);$$

i.e., $E(e^{atY})$ is the mgf of Y, evaluated at "at." Therefore,

$$m_W(t) = e^{bt}m_Y(at),$$

as claimed.

3.162. Suppose Y is a random variable with mgf $m_Y(t)$. Consider the function

$$r_Y(t) = \ln m_Y(t).$$

This is called the **cumulant generating function**; it is used in higher level probability courses. In this problem, we are asked to show

$$\begin{aligned} \left.\frac{d}{dt}r_Y(t)\right|_{t=0} &= E(Y) \\ \left.\frac{d^2}{dt^2}r_Y(t)\right|_{t=0} &= V(Y). \end{aligned}$$

Note that

$$\frac{d}{dt}r_Y(t) = \frac{d}{dt}\ln m_Y(t) = \frac{m'_Y(t)}{m_Y(t)},$$

where $m'_Y(t) = (d/dt)m_Y(t)$ is the derivative of $m_Y(t)$. Therefore,

$$\left.\frac{d}{dt}r_Y(t)\right|_{t=0} = \frac{m'_Y(0)}{m_Y(0)} = E(Y).$$

Note that $m_Y(0) = E(e^{0Y}) = E(1) = 1$. To show the second part, take another derivative:

$$\begin{aligned} \frac{d^2}{dt^2}r_Y(t) = \frac{d^2}{dt^2}\ln m_Y(t) &= \frac{d}{dt}\left[\frac{m'_Y(t)}{m_Y(t)}\right] \\ &= \frac{m''_Y(t)m_Y(t) - m'_Y(t)m'_Y(t)}{[m_Y(t)]^2}, \end{aligned}$$

by using the quotient rule for derivatives. Now, evaluate at $t = 0$, and you should see this reduces to

$$E(Y^2) - [E(Y)]^2 = V(Y)$$

because $m''_Y(0) = E(Y^2)$, $m'_Y(0) = E(Y)$, and $m_Y(0) = 1$.