3.48. Let $Y$ denote the number of radar sets detecting the missile; i.e., $Y \sim b(n, p = 0.9)$.

(a) Let $n = 5$; i.e., $Y \sim b(n = 5, p = 0.9)$. We have

$$P(Y = 4) = \binom{5}{4}(0.9)^4(0.1)^1 \approx 0.328$$

and

$$P(Y \geq 1) = 1 - P(Y = 0) = 1 - \binom{5}{0}(0.9)^0(0.1)^5 = 1 - 0.00001 = 0.99999.$$  

In R, these are calculated as

```r
> dbinom(4,5,0.9)
[1] 0.32805
> 1-pbinom(0,5,0.9)
[1] 0.99999
```

(b) Suppose $Y \sim b(n, p = 0.9)$. The missile will be detected when at least one radar set detects the missile; i.e., when the event $\{Y \geq 1\}$ occurs. The probability of this event is

$$P(Y \geq 1) = 1 - P(Y = 0) = 1 - \binom{n}{0}(0.9)^0(0.1)^n = 1 - (0.1)^n.$$  

We want this probability to be 0.999. Therefore,

$$1 - (0.1)^n = 0.999 \implies n = 3.$$  

We would need $n = 3$ radar sets to have this reliability level.

3.54. If $Y \sim b(n, p)$, then $Y$ counts the number of successes in $n$ Bernoulli trials. The random variable $Y^* = n - Y$ therefore counts the number of failures.

(a) From what I can tell, part (a) is obvious. The event $\{n - Y = y^*\}$ and $\{Y = n - y^*\}$ are the same event; i.e., just rewrite using algebra. Therefore, they have the same probability.

(b) For $y^* = 0, 1, 2, ..., n$, we have from part (a),

$$P(Y^* = y^*) = P(Y = n - y^*) = \binom{n}{n - y^*} p^{n-y^*} (1-p)^{n-(n-y^*)} = \binom{n}{y^*} (1-p)^y^* p^{n-y^*}.$$  

This shows that the number of failures $Y^* \sim b(n, 1-p)$. Note that $\binom{n}{y^*} = \binom{n}{n-y^*}$ from Exercise 2.68 (HW2).

(c) I don’t know that it is obvious, but it makes sense intuitively. Simply interchange the meaning of “success” and “failure.”

3.62. (a) We would have to assume that the events

$$A = \{\text{inspect plane that has a wing crack}\}$$

$$B = \{\text{inspect detail where crack located}\}$$

$$C = \{\text{detecting the damage}\}$$
are mutually independent with \( p_1 = P(A) \), \( p_2 = P(B) \), and \( p_3 = P(C) \). Detecting the crack would occur when \( A \cap B \cap C \) occurs. Under the mutually independence assumption,

\[
P(A \cap B \cap C) = P(A)P(B)P(C) = p_1p_2p_3.
\]

(b) Let \( Y \) denote the number of planes where a wing crack is detected. Then \( Y \sim b(n = 3, p = 0.36) \). Then

\[
P(Y \geq 1) = 1 - P(Y = 0) = 1 - \binom{3}{0} (0.36^0)(0.64)^3 \approx 1 - 0.262 = 0.738.
\]

In R,

\[
> 1-pbinom(0,3,0.36)
\]

\[
[1] 0.737856
\]

3.66. (a) Showing the geometric pmf sums to 1 was done in the notes; see pp 58.

(b) For \( y = 1, 2, 3, \ldots \), the pmf of \( Y \sim \text{geometric}(p) \) is \( p_Y(y) = q^{y-1}p \), where \( q = 1 - p \). Therefore, for \( y = 2, 3, 4, \ldots \),

\[
\frac{p_Y(y)}{p_Y(y-1)} = \frac{q^{y-1}p}{q^{(y-1)-1}p} = \frac{1}{q} = q,
\]

as claimed. Because \( q < 1 \), note that

\[
p_Y(y) = q p_Y(y-1) < p_Y(y-1),
\]

for \( y = 2, 3, 4, \ldots \). In other words, \( p_Y(1) > p_Y(2) > p_Y(3) > p_Y(4) > \cdots \). This means \( y = 1 \) is the most likely value in the geometric distribution; i.e., the mode of \( Y \) is \( y = 1 \).

3.71. (a) We are given that \( a \) is a positive integer. Using the complement rule, we have

\[
P(Y > a) = 1 - P(Y \leq a) = 1 - \sum_{y=1}^{a} q^{y-1}p \sum_{x=0}^{a-1} q^x = 1 - p \sum_{x=0}^{a-1} q^x.
\]

Note that

\[
\sum_{x=0}^{a} q^x = \frac{1 - q^a}{1 - q}
\]

because \( \sum_{x=0}^{a-1} q^x \) is a finite geometric sum with common ratio \( q \). Therefore, because \( 1 - q = p \), we have

\[
P(Y > a) = 1 - p \left( \frac{1 - q^a}{1 - q} \right) = 1 - (1 - q^a) = q^a,
\]

as claimed.

(b) Recall the definition of conditional probability and write

\[
P(Y > a + b | Y > a) = \frac{P(Y > a + b \text{ and } Y > a)}{P(Y > a)} = \frac{P(Y > a + b)}{P(Y > a)}.
\]

The last step is true because \( \{ Y > a + b \} \subset \{ Y > a \} \) so that \( \{ Y > a + b \} \cap \{ Y > a \} = \{ Y > a + b \} \). Therefore,

\[
P(Y > a + b | Y > a) = \frac{P(Y > a + b)}{P(Y > a)} = \frac{q^{a+b}}{q^a} = q^b.
\]

---

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This shows that
\[ P(Y > a + b|Y > a) = P(Y > b) \]
which is a condition known as the memoryless condition. The geometric random variable is the only discrete random variable that satisfies this property.

**Interpretation:** Suppose Experimenter 1 is observing Bernoulli trials, and the first success has not occurred in the first \(a\) trials. This is what is meant by the “given” event \(\{Y > a\}\). The probability she has to wait an additional \(b\) trials to observe the first success; i.e., \(\{Y > a + b\}\) is the same as for another experimenter, say Experimenter 2, having to wait \(b\) trials from the outset. In other words, the fact that Experimenter 1 has not observed a success in the first \(a\) trials has been “forgotten.”

(c) I don’t know that this is obvious, but it is certainly true. I think the key is that we are waiting for the “first success.” Because the trials are independent, observing a bunch of failures from the outset doesn’t affect future trials and hence does not impact when we will observe the first success.

**3.77.** Suppose \(Y \sim \text{geometric}(p)\). We want to calculate
\[
P(Y = \text{odd integer}) = P(Y = 1) + P(Y = 3) + P(Y = 5) + P(Y = 7) + \cdots
\]
\[
= p + q^2p + q^4p + q^6p + \cdots
\]
\[
= p\left(1 + q^2 + q^4 + q^6 + \cdots\right) = p \sum_{j=0}^{\infty} (q^2)^j.
\]

Note that \(\sum_{j=0}^{\infty} (q^2)^j\) is an infinite geometric sum with common ratio \(q^2\). Therefore,
\[
\sum_{j=0}^{\infty} (q^2)^j = \frac{1}{1-q^2}.
\]
The result follows immediately.

**3.97.** In this problem, we envision each oil well as a “trial,” where “success” means that the well produces oil (i.e., “strikes” oil). Assume the oil wells are independent, each with probability of success \(p = 0.2\). These are the assumptions needed for the question in part (c); i.e., the Bernoulli trial assumptions hold.

(a) Let \(Y\) denote the number of wells observed to find the first productive well (i.e., the first success). Then \(Y \sim \text{geometric}(p = 0.2)\) and
\[
P(Y = 3) = (1 - 0.2)^2(0.2) = 0.128.
\]
In R,
\[
> \text{dgeom}(3-1,0.2)
\]
[1] 0.128
(b) Let $X$ denote the number of wells to find the third productive well (i.e., the third success). Then $X \sim \text{nib}(r = 3, p = 0.2)$ and 

$$P(X = 7) = \binom{7-1}{3-1}(0.2)^3(1-0.2)^4 \approx 0.049.$$ 

In R, 

```r
> dnbinom(7-3,3,0.2)
[1] 0.049152
```

(c) See discussion above.

(d) In this part, we want $E(X)$ and $V(X)$ in part (b). With $r = 3$ and $p = 0.2$, we have 

$$E(X) = \frac{r}{p} = \frac{3}{0.2} = 15 \text{ wells}. $$

Also, 

$$V(X) = \frac{r}{p^2} = \frac{3(0.8)}{0.2^2} = 60 \text{ (wells)}^2.$$ 

3.159. In Exercise 3.158, $Y$ is a random variable with mgf $m_Y(t)$. The random variable $W = aY + b$ is a linear function of $Y$. You showed in HW4 that $m_W(t) = e^{bt}m_Y(at)$. To find $E(W)$ note that 

$$\frac{d}{dt}m_W(t) = \frac{d}{dt}e^{bt}m_Y(at) = be^{bt}m_Y(at) + e^{bt}m'_Y(at) \times a = e^{bt} \left[ bm_Y(at) + am'_Y(at) \right].$$

Evaluating this derivative at $t = 0$ gives 

$$E(W) = e^0 \left[ bm_Y(0) + am'_Y(0) \right] = aE(Y) + b.$$ 

Above we used the fact that $m_Y(0) = 1$ and $m'_Y(0) = E(Y)$.

To find $V(W)$, we can find $E(W^2)$ first. Taking another derivative, we have 

$$\frac{d^2}{dt^2}m_W(t) = \frac{d}{dt} \left\{ e^{bt} \left[ bm_Y(at) + am'_Y(at) \right] \right\} = be^{bt} \left[ bm_Y(at) + am'_Y(at) \right] + e^{bt} \left[ abm'_Y(at) + a^2m''_Y(at) \right].$$

Evaluating this derivative at $t = 0$ gives 

$$E(W^2) = be^0 \left[ bm_Y(0) + am'_Y(0) \right] + e^0 \left[ abm'_Y(0) + a^2m''_Y(0) \right] = b[bE(Y)] + abE(Y) + a^2E(Y^2) = a^2E(Y^2) + 2abE(Y) + b^2.$$ 

From the variance computing formula, we have 

$$V(W) = E(W^2) - [E(W)]^2 = a^2E(Y^2) + 2abE(Y) + b^2 - [aE(Y) + b]^2 = a^2E(Y^2) + 2abE(Y) + b^2 - \{a^2[E(Y)]^2 + 2abE(Y) + b^2\} = a^2E(Y^2) - a^2[E(Y)]^2 = a^2\{E(Y^2) - [E(Y)]^2\} = a^2V(Y)$$

as claimed.
3.160. We are given that $Y \sim \text{b}(n, p)$. Recall $Y$ counts the number of successes in $n$ Bernoulli trials. Therefore, $Y^* = n - Y$ counts the number of failures. We know $E(Y) = np$ and $V(Y) = npq$, where $q = 1 - p$.

(a) We have

$$E(Y^*) = E(n - Y) = n - E(Y) = n - np = n(1 - p) = nq$$

and

$$V(Y^*) = V(n - Y) = (−1)^2V(Y) = npq.$$ 

(b) The mgf of $Y^*$ is

$$m_{Y^*}(t) = E(e^{tY^*}) = E[e^{t(n-1)}] = e^{nt}E(e^{-tY}) = e^{nt}m_Y(-t) = e^{nt}(q + pe^{-t})^n = (e^t)^n(q + pe^{-t})^n = (qe^t + p)^n.$$ 

(c) The mgf in part (b) is the mgf of a binomial distribution with number of trials $n$ and “success probability” $q = 1 - p$. Therefore, $Y^* \sim \text{b}(n, 1 - p)$.

(d) $Y^* = n - Y$ counts the number of failures.

(e) I already answered this in another problem.

3.188. We are given that $Y \sim \text{b}(n, p)$. Use the definition of conditional probability; i.e.,

$$P(Y > 1 | Y \geq 1) = \frac{P(Y > 1 \text{ and } Y \geq 1)}{P(Y \geq 1)} = \frac{P(Y > 1)}{P(Y \geq 1)}.$$ 

The last step is true because $\{Y > 1\} \subset \{Y \geq 1\}$ so that $\{Y > 1\} \cap \{Y \geq 1\} = \{Y > 1\}$. Now use the complement rule and write

$$P(Y > 1) = 1 - P(Y \leq 1) = 1 - P(Y = 0) - P(Y = 1) = 1 - \binom{n}{0}p^0(1-p)^{n-0} - \binom{n}{1}p^1(1-p)^{n-1} = 1 - (1 - p)^n - np(1-p)^{n-1}.$$ 

Also,

$$P(Y \geq 1) - 1 - P(Y = 0) = 1 - \binom{n}{0}p^0(1-p)^n = 1 - (1 - p)^n.$$ 

The result follows.