4.43. We are given that a circle's radius $R \sim \mathcal{U}(0,1)$. Therefore, the pdf of R is

$$f_R(r) = \begin{cases} 1, & 0 \le r \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

The area of the circle is $A = \pi R^2$. The mean of A is $E(A) = E(\pi R^2) = \pi E(R^2)$. The second moment of R is

$$E(R^2) = \int_0^1 r^2 f_R(r)dr = \int_0^1 r^2 dr = \left(\frac{r^3}{3}\right)\Big|_0^1 = \frac{1}{3}.$$

Therefore,

$$E(A) = \pi E(R^2) = \frac{\pi}{3}.$$

The variance of A is $V(A) = V(\pi R^2) = \pi^2 V(R^2)$. How do we calculate $V(R^2)$? Use the variance computing formula for the "random variable" R^2; i.e.,

$$V(R^2) = E(R^4) - [E(R^2)]^2 = E(R^4) - \left(\frac{1}{3}\right)^2.$$

The fourth moment of R is

$$E(R^4) = \int_0^1 r^4 f_R(r)dr = \int_0^1 r^4 dr = \left(\frac{r^5}{5}\right)\Big|_0^1 = \frac{1}{5}.$$

Therefore,

$$V(A) = \pi^2 V(R^2) = \pi^2\left[\frac{1}{5} - \left(\frac{1}{3}\right)^2\right] = \frac{4\pi^2}{45}.$$

Note: You could also get $E(R^2)$ and $E(R^4)$ using the mgf of $R \sim \mathcal{U}(0,1)$; i.e.,

$$m_R(t) = \frac{e^t - 1}{t}, \quad t \neq 0;$$

i.e., calculate 2nd and 4th derivatives and evaluate at $t = 0$. You should get the same answers for $E(R^2)$ and $E(R^4)$.

4.44. (a) The change in depth $Y \sim \mathcal{U}(-2,2)$; i.e., $\theta_1 = -2$ and $\theta_2 = 2$. Therefore, the pdf of Y is

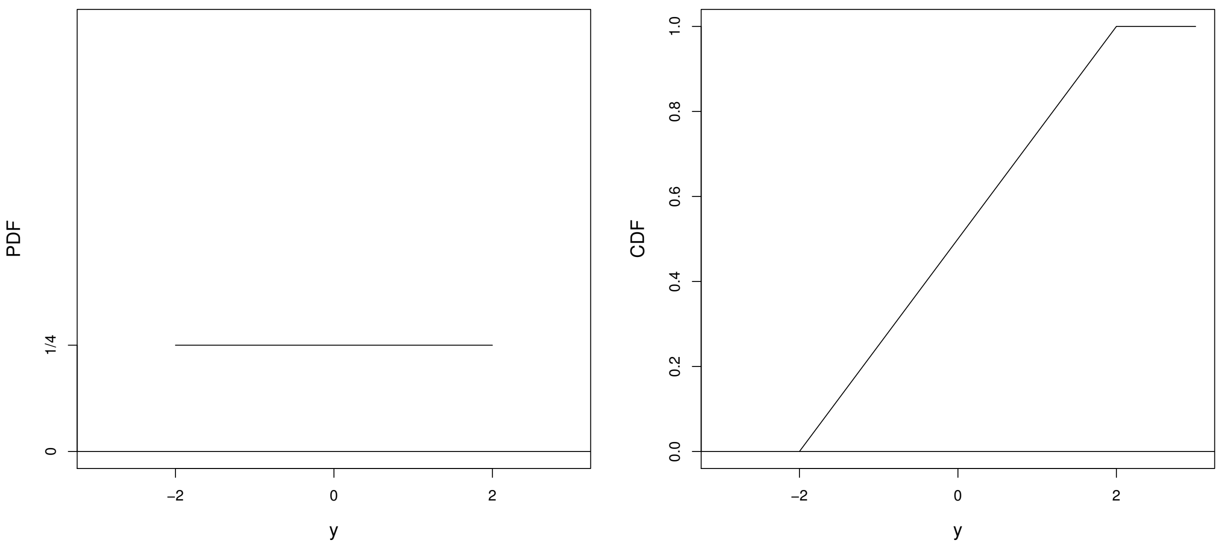
$$f_Y(y) = \begin{cases} \frac{1}{4}, & -2 < y \le 2 \\ 0, & \text{otherwise.} \end{cases}$$

(b) The general expression for the cdf of Y is

$$F_Y(y) = \int_{-\infty}^{y} f_Y(t)dt,$$

which we must calculate for all $y \in \mathbb{R}$.
Case 1: When $y \le -2$,

$$F_Y(y) = \int_{-\infty}^{y} f_Y(t)dt = \int_{-\infty}^{y} 0dt = 0.$$



Case 2: When $-2 < y < 2$,

$$F_Y(y) = \int_{-\infty}^{y} f_Y(t)dt = \int_{-\infty}^{-2} 0dt + \int_{-2}^{y} \frac{1}{4}dt = 0 + \frac{t}{4}\bigg|_{-2}^{y} = \frac{y-(-2)}{4} = \frac{y+2}{4}.$$

Case 3: When $y \geq 2$,

$$F_Y(y) = \int_{-\infty}^{y} f_Y(t)dt = \int_{-\infty}^{-2} 0dt + \underbrace{\int_{-2}^{2} \frac{t}{4}dt}_{=1} + \int_{2}^{y} 0dt = 1.$$

Summarizing, the cdf of Y is

$$F_Y(y) = \begin{cases} 0, & y \leq -2 \\ \dfrac{y+2}{4}, & -2 < y < 2 \\ 1, & y \geq 2. \end{cases}$$

The pdf and cdf of Y are shown side by side in the figure above. Here is the R code I used:

```
y = seq(-2,2,0.01)
pdf = (1/4)+y-y
# Plot PDF
plot(y,pdf,xlab="y",ylab="PDF",type="l",xaxt="n",yaxt="n",xlim=c(-3,3),ylim=c(0,1),
    cex.lab=1.25)
abline(h=0)
axis(1,at=c(-2,0,2),labels = c("-2","0","2"))
axis(2,at=c(0,1/4),labels = c("0","1/4"))
```

```
# Plot CDF
cdf = (y+2)/4
plot(y,cdf,xlab="y",ylab="CDF",type="l",xaxt="n",xlim=c(-3,3),ylim=c(0,1),
    cex.lab=1.25)
abline(h=0)
lines(c(2,3),c(1,1),lty=1)
axis(1,at=c(-2,0,2),labels = c("-2","0","2"))
```

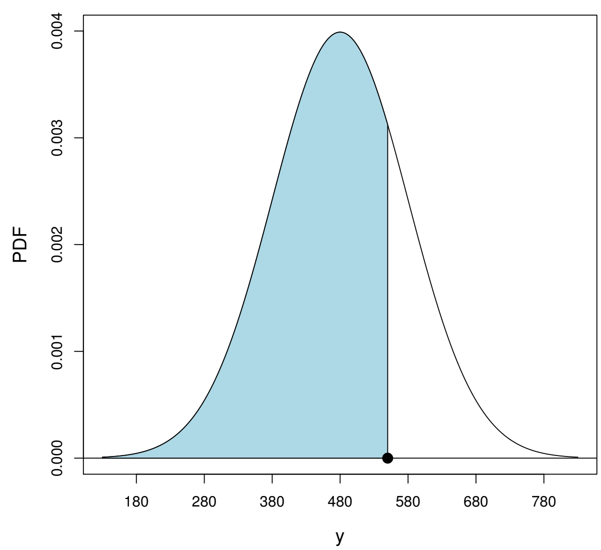
4.77. Let Y denote the SAT mathematics exam score. We assume $Y \sim \mathcal{N}(\mu = 480, \sigma^2 = 100^2)$. In part (a), we want to calculate

$$P(Y < 550) = \int_{-\infty}^{550} \underbrace{\frac{1}{\sqrt{2\pi}(100)} e^{-\frac{1}{2}\left(\frac{y-480}{100}\right)^2}}_{\mathcal{N}(480,100^2)\text{ pdf}} dy \;\; = \;\; F_Y(550)$$

where $F_Y(\cdot)$ denotes the $\mathcal{N}(\mu = 480, \sigma^2 = 100^2)$ cdf. In R, this is calculated as

```
> pnorm(550,480,100)
[1] 0.7580363
```

Therefore, about 76 percent of the students will score 550 or below. Here is the pdf of Y with $P(Y < 550)$ shaded:



Here is the R code I used to make the figure above:

```
y = seq(130,830,0.1)
pdf = dnorm(y,480,100)
plot(y,pdf,type="l",xlab="y",ylab="PDF",xaxp=c(180,780,6),cex.lab=1.25)
abline(h=0)
# Add shading corresponding to P(Y<550)
x = seq(130,550,0.1)
y = dnorm(x,480,100)
polygon(c(130,x,550),c(0,y,0),col="lightblue")
points(x=550,y=0,pch=19,cex=1.5)
```

(b) In part (a), we determined that 550 is the 0.7580363 quantile of the $\mathcal{N}(\mu = 480, \sigma^2 = 100^2)$ distribution; i.e., $P(Y < 550) = 0.7580363$. In part (b), we want to find this same quantile from the distribution of $X \sim \mathcal{N}(\mu = 18, \sigma^2 = 6^2)$; i.e., the distribution of X = ACT math score. From R,

```
> qnorm(0.7580363,18,6)
[1] 22.2
```

4.79. In this problem, we want to show the $\mathcal{N}(\mu, \sigma^2)$ pdf $f_Y(y)$ has points of inflection at $y = \mu \pm \sigma$. We can do this by finding the second derivative of $f_Y(y)$. The first derivative of $f_Y(y)$ is

$$\begin{aligned} f'_Y(y) = \frac{d}{dy} f_Y(y) &= \frac{d}{dy}\left[\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}\right] \\ &= \frac{1}{\sqrt{2\pi}\sigma}\,\frac{d}{dy}\left[e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}\right] \\ &= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2} \times 2\left(-\frac{1}{2}\right)\left(\frac{y-\mu}{\sigma}\right)^1\left(\frac{1}{\sigma}\right) \;=\; -\frac{(y-\mu)}{\sqrt{2\pi}\sigma^3} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}. \end{aligned}$$

To find the second derivative, use the product rule:

$$\begin{aligned} f''_Y(y) = \frac{d^2}{dy^2} f_Y(y) &= \frac{d}{dy}\left[-\frac{(y-\mu)}{\sqrt{2\pi}\sigma^3} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}\right] \\ &= -\frac{1}{\sqrt{2\pi}\sigma^3} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2} + \left[-\frac{(y-\mu)}{\sqrt{2\pi}\sigma^3}\right] e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2} \times 2\left(-\frac{1}{2}\right)\left(\frac{y-\mu}{\sigma}\right)^1\left(\frac{1}{\sigma}\right) \\ &= -\frac{1}{\sqrt{2\pi}\sigma^3} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2} + \frac{(y-\mu)^2}{\sqrt{2\pi}\sigma^5} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2} \\ &= \frac{1}{\sqrt{2\pi}\sigma^3} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}\left[\left(\frac{y-\mu}{\sigma}\right)^2 - 1\right]. \end{aligned}$$

Now, set $f''_Y(y)$ equal to zero and solve for y; we get

$$\underbrace{\frac{1}{\sqrt{2\pi}\sigma^3} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}}_{\text{this can never be 0}}\left[\left(\frac{y-\mu}{\sigma}\right)^2 - 1\right] \stackrel{\text{set}}{=} 0 \implies \left(\frac{y-\mu}{\sigma}\right)^2 - 1 = 0$$

$$\implies \left(\frac{y-\mu}{\sigma}\right)^2 = 1 \implies \frac{y-\mu}{\sigma} = \pm 1.$$

Note that if $(y-\mu)/\sigma = +1$, then

$$y - \mu = \sigma \implies y = \mu + \sigma.$$

If $(y-\mu)/\sigma = -1$, then

$$y - \mu = -\sigma \implies y = \mu - \sigma.$$

This argument shows the points of inflection on the $\mathcal{N}(\mu, \sigma^2)$ pdf $f_Y(y)$ are at $y = \mu \pm \sigma$. For example, the points of inflection on the $\mathcal{N}(\mu = 480, \sigma^2 = 100^2)$ pdf in Exercise 4.77 are $y = 380$ and $y = 580$.

4.81. The gamma function, which is defined for $\alpha > 0$, is

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1}e^{-y}dy.$$

In part (a), put in $\alpha = 1$; we get

$$\Gamma(1) = \int_0^\infty y^{1-1}e^{-y}dy = \int_0^\infty e^{-y}dy = -e^{-y}\Big|_0^\infty = -\left(\lim_{y\to\infty} e^{-y} - 1\right) = -(0-1) = 1.$$

In part (b), suppose $\alpha > 1$. In the integral

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1}e^{-y}dy,$$

use integration by parts with

$$\begin{aligned} u &= y^{\alpha-1} & du &= (\alpha-1)y^{\alpha-2}dy \\ dv &= e^{-y} & v &= -e^{-y}. \end{aligned}$$

With these selections,

$$\begin{aligned} \Gamma(\alpha) = \int_0^\infty y^{\alpha-1}e^{-y}dy &= \underbrace{-y^{\alpha-1}e^{-y}\Big|_0^\infty}_{=0} + (\alpha-1)\int_0^\infty y^{\alpha-2}e^{-y}dy \\ &= (\alpha-1)\underbrace{\int_0^\infty y^{(\alpha-1)-1}e^{-y}dy}_{=\Gamma(\alpha-1)} = (\alpha-1)\Gamma(\alpha-1), \end{aligned}$$

as claimed.

4.92. The time to complete the operation is a random variable $Y \sim$ exponential($\beta = 10$). Recall that $E(Y) = \beta = 10$ and $V(Y) = \beta^2 = 100$. Define $C = 100 + 40Y + 3Y^2$. We want to find $E(C)$ and $V(C)$. Finding $E(C)$ is easy. Note that

$$E(C) = E(100 + 40Y + 3Y^2) = 100 + 40E(Y) + 3E(Y^2).$$

We can get the second moment of Y quickly from the variance computing formula. Recall

$$V(Y) = E(Y^2) - [E(Y)]^2 \implies E(Y^2) = V(Y) + [E(Y)]^2.$$

Therefore,

$$E(Y^2) = 100 + (10)^2 = 200$$

and

$$E(C) = 100 + 40(10) + 3(200) = 1100.$$

Getting $V(C)$ is harder. One way is to use the variance computing formula for the function C; i.e.,

$$V(C) = E(C^2) - [E(C)]^2 = E(C^2) - (1100)^2.$$

We now have to get $E(C^2)$. We have

$$\begin{aligned} E(C^2) &= E[(100+40Y+3Y^2)^2] \\ &= E(10000+1600Y^2+9Y^4+8000Y+600Y^2+240Y^3) \\ &= 10000+8000E(Y)+2200E(Y^2)+240E(Y^3)+9E(Y^4). \end{aligned}$$

We know $E(Y)=10$ and $E(Y^2)=200$. We have to get the third and fourth moments $E(Y^3)$ and $E(Y^4)$, respectively. We could get these using the moment generating function, but it might be easier to just calculate the moments directly. Note that

$$E(Y^3) = \int_0^\infty y^3 \frac{1}{10} e^{-y/10} dy = \frac{1}{10}\int_0^\infty y^{4-1} e^{-y/10} dy = \frac{1}{10}\Gamma(4)10^4 = 6000.$$

Also,

$$E(Y^4) = \int_0^\infty y^4 \frac{1}{10} e^{-y/10} dy = \frac{1}{10}\int_0^\infty y^{5-1} e^{-y/10} dy = \frac{1}{10}\Gamma(5)10^5 = 240000.$$

Therefore,

$$E(C^2) = 10000+8000(10)+2200(200)+240(6000)+9(240000) = 4130000.$$

Finally,

$$V(C) = E(C^2)-(1100)^2 = 4130000-(1100)^2 = 2920000.$$

Note: Another way to get $V(C)$ is to use the definition of variance for the random variable C. Recall that

$$\begin{aligned} V(C)=E[(C-1100)^2] &= E[(100+40Y+3Y^2-1100)^2] \\ &= \int_0^\infty (100+40y+3y^2-1100)^2 \, \frac{1}{10} e^{-y/10} dy. \end{aligned}$$

You can do this integral numerically in R:

```
> integrand <- function(y){(100+40*y+3*y^2-1100)^2*(1/10)*exp(-y/10)}
> integrate(integrand,lower=0,upper=Inf)
2920000 with absolute error < 19
```

4.94. Let Y denote the one-hour CO concentration (measured in ppm). In part (a), we assume $Y \sim$ exponential($\beta = 3.6$) and want to calculate $P(Y>9)$. The easiest way is to note that

$$P(Y>9) = 1-P(Y\le 9) = 1-F_Y(9),$$

where $F_Y(\cdot)$ is the cdf of Y. Recall the cdf of $Y \sim$ exponential(β) is given by

$$F_Y(y) = \begin{cases} 0, & y \le 0 \\ 1-e^{-y/\beta}, & y > 0. \end{cases}$$

Therefore,

$$P(Y>9) = 1-F_Y(9) = 1-\left(1-e^{-9/3.6}\right) = e^{-9/3.6} \approx 0.082.$$

If you didn't remember to use the cdf here, you could always just calculate $P(Y > 9)$ directly using the pdf:

$$\begin{aligned} P(Y > 9) = \int_9^\infty \frac{1}{3.6} e^{-y/3.6} dy &= \frac{1}{3.6}\left(-3.6 e^{-y/3.6}\Big|_9^\infty \right) \\ &= e^{-y/3.6}\Big|_\infty^9 = e^{-9/3.6} - \underbrace{\lim_{y\to\infty} e^{-y/3.6}}_{=\,0} = e^{-9/3.6} \approx 0.082. \end{aligned}$$

You can also use R to check your work:

```
> 1-pexp(9,1/3.6)
[1] 0.082085
```

In part (b), we perform the same calculation but with $\beta = 2.5$ ppm instead:

$$P(Y > 9) = 1 - \left(1 - e^{-9/2.5}\right) = e^{-9/2.5} \approx 0.027.$$

4.110. The support of Y is $R = \{y : y > 0\}$, so this makes me think of the gamma family of distributions. Look at the kernel of the pdf:

$$y^2 e^{-2y} = y^{3-1} e^{-y/(1/2)}.$$

This is the kernel of the gamma distribution with $\alpha = 3$ and $\beta = 1/2$. Sure enough, the constant out front in the gamma pdf

$$\frac{1}{\Gamma(3)\left(\frac{1}{2}\right)^3} = \frac{1}{2\left(\frac{1}{8}\right)} = 4.$$

Therefore, $Y \sim \text{gamma}(\alpha = 3, \beta = 1/2)$. We have

$$E(Y) = \alpha\beta = \frac{3}{2}$$

and

$$V(Y) = \alpha\beta^2 = \frac{3}{4}.$$

4.111. Suppose $Y \sim \text{gamma}(\alpha, \beta)$. The hardest part about this problem is keeping a and α separate (the authors should have used another symbol other than a here). In part (a), suppose $\alpha + a > 0$. From the definition of mathematical expectation,

$$\begin{aligned} E(Y^a) = \int_0^\infty y^a \underbrace{\frac{1}{\Gamma(\alpha)\beta^\alpha} y^{\alpha-1} e^{-y/\beta}}_{\text{gamma}(\alpha,\beta)\text{ pdf}} dy &= \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} y^{\alpha+a-1} e^{-y/\beta} dy \\ &= \frac{1}{\Gamma(\alpha)\beta^\alpha} \underbrace{\int_0^\infty y^{\alpha+a-1} e^{-y/\beta} dy}_{=\,\Gamma(a+\alpha)\beta^{\alpha+a}} \\ &= \frac{1}{\Gamma(\alpha)\beta^\alpha} \Gamma(a+\alpha)\beta^{\alpha+a} = \frac{\beta^a \Gamma(\alpha+a)}{\Gamma(\alpha)}, \end{aligned}$$

as claimed.

(b) When we say

$$\int_0^\infty y^{\alpha+a-1}e^{-y/\beta}dy = \Gamma(\alpha+a)\beta^{\alpha+a}$$

in part (a), we are using the fact that

$$\int_0^\infty \frac{1}{\Gamma(\alpha+a)\beta^{\alpha+a}}y^{\alpha+a-1}e^{-y/\beta}dy = 1,$$

that is, the gamma$(\alpha+a,\beta)$ pdf integrates to 1. Recall that the shape parameter in the gamma family must be strictly larger than zero. Therefore, we must require $\alpha+a>0$ to make this argument.
(c) When $a=1$, we have

$$E(Y) = \frac{\beta^1\Gamma(\alpha+1)}{\Gamma(\alpha)} = \beta\frac{\alpha\Gamma(\alpha)}{\Gamma(\alpha)} = \alpha\beta.$$

(d) Note that

$$E(\sqrt{Y}) = E(Y^a),$$

when $a=\frac{1}{2}$. Therefore,

$$E(\sqrt{Y}) = E(Y^{\frac{1}{2}}) = \frac{\beta^{\frac{1}{2}}\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma(\alpha)}.$$

We must require $\alpha+a=\alpha+\frac{1}{2}>0$.

(e)

- Note that

  $$E\left(\frac{1}{Y}\right) = E(Y^a),$$

  when $a=-1$. Therefore,

  $$E\left(\frac{1}{Y}\right) = \frac{\beta^{-1}\Gamma\left(\alpha-1\right)}{\Gamma(\alpha)} = \frac{\beta^{-1}\Gamma\left(\alpha-1\right)}{(\alpha-1)\Gamma\left(\alpha-1\right)} = \frac{1}{(\alpha-1)\beta}.$$

  We must require $\alpha+a=\alpha-1>0 \Longleftrightarrow \alpha>1$. **Remark:** $E(\frac{1}{Y})$ is called the **first inverse moment** of a random variable Y.

- Note that

  $$E\left(\frac{1}{\sqrt{Y}}\right) = E(Y^a),$$

  when $a=-\frac{1}{2}$. Therefore,

  $$E\left(\frac{1}{\sqrt{Y}}\right) = \frac{\beta^{-\frac{1}{2}}\Gamma\left(\alpha-\frac{1}{2}\right)}{\Gamma(\alpha)}.$$

  We must require $\alpha+a=\alpha-\frac{1}{2}>0 \Longleftrightarrow \alpha>\frac{1}{2}$.

- Note that

  $$E\left(\frac{1}{Y^2}\right) = E(Y^a),$$

when $a = -2$. Therefore,

$$E\left(\frac{1}{Y^2}\right) = \frac{\beta^{-2}\Gamma(\alpha-2)}{\Gamma(\alpha)} = \frac{\beta^{-2}\Gamma(\alpha-2)}{(\alpha-1)(\alpha-2)\Gamma(\alpha-2)} = \frac{1}{(\alpha-1)(\alpha-2)\beta^2}.$$

We must require $\alpha + a = \alpha - 2 > 0 \Longleftrightarrow \alpha > 2$. **Remark:** $E(\frac{1}{Y^2})$ is called the **second inverse moment** of a random variable Y.

4.140. (a) We recognize

$$m_Y(t) = \left(\frac{1}{1-4t}\right)^2$$

as the mgf of a gamma distribution with $\alpha = 2$ and $\beta = 4$ (provided that $t < \frac{1}{4}$). Therefore, if Y has this mgf, then $Y \sim \text{gamma}(\alpha = 2, \beta = 4)$.
(b) We recognize

$$m_Y(t) = \frac{1}{1-3.2t}$$

as the mgf of an exponential distribution with $\beta = 3.2$ (provided that $t < 1/3.2$). Therefore, if Y has this mgf, then $Y \sim \text{exponential}(\beta = 3.2)$.
(c) We recognize

$$m_Y(t) = \exp\left(-5t + 6t^2\right) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)$$

as the mgf of a normal distribution with mean $\mu = -5$ and

$$6 = \frac{\sigma^2}{2} \implies \sigma^2 = 12.$$

Therefore, if Y has this mgf, then $Y \sim \mathcal{N}(\mu = -5, \sigma^2 = 12)$.