4.128. Let Y denote the weekly repair cost. Note that the nonzero part of the pdf $f_Y(y) = 3(1-y)^2$ is what results in the beta(α, β) family when $\alpha = 1$ and $\beta = 3$; observe

$$\frac{\Gamma(1+3)}{\Gamma(1)\Gamma(3)}\, y^{1-1}(1-y)^{3-1} = 3(1-y)^2.$$

Therefore $Y \sim \text{beta}(1,3)$. We want to find $\phi_{0.9}$, the $p = 0.9$ quantile of this distribution. Note that $\phi_{0.9}$ solves

$$P(Y \leq \phi_{0.9}) = 0.9 \iff P(Y > \phi_{0.9}) = 0.1;$$

i.e., the "cost will exceed....only 10% of the time." We can find $\phi_{0.9}$ by solving

$$\begin{aligned}
0.9 \overset{\text{set}}{=} \int_0^{\phi_{0.9}} 3(1-y)^2 dy &= -\int_1^{1-\phi_{0.9}} 3u^2 du \qquad (u = 1-y) \\
&= -u^3 \Big|_1^{1-\phi_{0.9}} \\
&= -\left[(1-\phi_{0.9})^3 - 1\right] \\
&\implies (1-\phi_{0.9})^3 = 0.1 \implies \phi_{0.9} = 1 - (0.1)^{1/3} \approx 0.536.
\end{aligned}$$

We could check our work in R using the `qbeta` function:

```
> qbeta(0.9,1,3) # p=0.9 quantile
[1] 0.5358411
```

Therefore, we would set the weekly budget at approximately 53.6 dollars; this would lead to exceeding the budget only about 10 percent of the time.

4.129. Let Y denote the proportion of time the machine is down; $Y \sim \text{beta}(1,2)$. Let

$$C = 10 + 20Y + 4Y^2$$

denote the cost of machine down time. The mean of C is

$$E(C) = E(10 + 20Y + 4Y^2) = 10 + 20E(Y) + 4E(Y^2).$$

The beta mgf is not in a friendly form, so let's not use it. We derived formulas for the mean and variance of a beta distribution; i.e.,

$$E(Y) = \frac{1}{1+2} = \frac{1}{3} \qquad \text{and} \qquad V(Y) = \frac{1(2)}{(1+2)^2(1+2+1)} = \frac{2}{36} = \frac{1}{18}.$$

Therefore, the second moment of Y is

$$E(Y^2) = V(Y) + [E(Y)]^2 = \frac{1}{18} + \left(\frac{1}{3}\right)^2 = \frac{1}{6}.$$

Finally,

$$E(C) = 10 + 20E(Y) + 4E(Y^2) = 10 + 20\left(\frac{1}{3}\right) + 4\left(\frac{1}{6}\right) = \frac{52}{3} \approx 17.33.$$

Therefore, the expected (mean) cost due to down time is approximately $1,733.

Note: We could have calculated $E(C) = E(10 + 20Y + 4Y^2)$ from first principles by writing

$$E(10 + 20Y + 4Y^2) = \int_0^1 (10 + 20y + 4y^2) \times 2(1 - y)dy$$

and then doing this integral. We would get the same answer. In fact, in R,

```
> integrand = function(y) {(10+20*y+4*y^2)*2*(1-y)}
> integrate(integrand,0,1)
17.33333 with absolute error < 1.9e-13
```

Getting the variance $V(C)$ is harder if you are doing things by hand. An easy way you could do this is to write

$$\begin{aligned} V(C) = E[(C - \mu_C)^2] = E\left[\left(C - \frac{52}{3}\right)^2\right] &= E\left[\left(10 + 20Y + 4Y^2 - \frac{52}{3}\right)^2\right] \\ &= \int_0^1 \left(10 + 20y + 4y^2 - \frac{52}{3}\right)^2 \times 2(1 - y)dy. \end{aligned}$$

This integral can be calculated numerically in R:

```
> integrand.2 = function(y) {(10+20*y+4*y^2-(52/3))^2*2*(1-y)}
> integrate(integrand.2,0,1)
29.95556 with absolute error < 3.3e-13
```

Otherwise, we can do it as follows (the long way). First write

$$V(C) = E(C^2) - [E(C)]^2 = E(C^2) - \left(\frac{52}{3}\right)^2.$$

Now, we have to get $E(C^2)$, the second moment of C. Note that

$$\begin{aligned} C^2 = (10 + 20Y + 4Y^2)^2 &= 100 + 400Y^2 + 16Y^4 + 400Y + 80Y^2 + 160Y^3 \\ &= 100 + 400Y + 480Y^2 + 160Y^3 + 16Y^4. \end{aligned}$$

Therefore,

$$E(C^2) = 100 + 400E(Y) + 480E(Y^2) + 160E(Y^3) + 16E(Y^4).$$

We already know $E(Y) = \frac{1}{3}$ and $E(Y^2) = \frac{1}{6}$. The third moment of $Y \sim \text{beta}(1, 2)$ is

$$E(Y^3) = \int_0^1 y^3 \times 2(1 - y)dy = 2\int_0^1 \underbrace{y^{4-1}(1 - y)^{2-1}}_{\text{beta(4,2) kernel}} dy = \frac{2\Gamma(4)\Gamma(2)}{\Gamma(6)} = \frac{1}{10}.$$

Similarly,

$$E(Y^4) = \int_0^1 y^4 \times 2(1 - y)dy = 2\int_0^1 \underbrace{y^{5-1}(1 - y)^{2-1}}_{\text{beta(5,2) kernel}} dy = \frac{2\Gamma(5)\Gamma(2)}{\Gamma(7)} = \frac{48}{720}.$$

Therefore,

$$E(C^2) = 100 + 400\left(\frac{1}{3}\right) + 480\left(\frac{1}{6}\right) + 160\left(\frac{1}{10}\right) + 16\left(\frac{48}{720}\right) = 330.4.$$

Finally,

$$V(C) = E(C^2) - [E(C)]^2 = 330.4 - \left(\frac{52}{3}\right)^2 \approx 29.96 \text{ (100s dollars)}^2.$$

4.130. Here we are being asked to show

$$V(Y) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)},$$

where $Y \sim \text{beta}(\alpha, \beta)$. In class, we derived

$$E(Y) = \frac{\alpha}{\alpha+\beta}.$$

Recall that

$$V(Y) = E(Y^2) - [E(Y)]^2 = E(Y^2) - \left(\frac{\alpha}{\alpha+\beta}\right)^2.$$

Therefore, all we have to do is to derive the second moment $E(Y^2)$ and then do some algebra. Note that

$$\begin{aligned} E(Y^2) = \int_0^1 y^2 \times \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\, y^{\alpha-1}(1-y)^{\beta-1}dy &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \underbrace{y^{\alpha+2-1}(1-y)^{\beta-1}}_{\text{beta}(\alpha+2,\beta)\text{ kernel}}\, dy \\ &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+2)\Gamma(\beta)}{\Gamma(\alpha+\beta+2)}. \end{aligned}$$

Now use the recursive property of the gamma function to write

$$\Gamma(\alpha+2) = (\alpha+1)\Gamma(\alpha+1) = (\alpha+1)\alpha\Gamma(\alpha)$$

and

$$\Gamma(\alpha+\beta+2) = (\alpha+\beta+1)\Gamma(\alpha+\beta+1) = (\alpha+\beta+1)(\alpha+\beta)\Gamma(\alpha+\beta).$$

Therefore, $E(Y^2)$ above becomes

$$E(Y^2) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{(\alpha+1)\alpha\Gamma(\alpha)\Gamma(\beta)}{(\alpha+\beta+1)(\alpha+\beta)\Gamma(\alpha+\beta)} = \frac{(\alpha+1)\alpha}{(\alpha+\beta+1)(\alpha+\beta)}.$$

Therefore,

$$\begin{aligned} V(Y) = E(Y^2) - [E(Y)]^2 &= \frac{(\alpha+1)\alpha}{(\alpha+\beta+1)(\alpha+\beta)} - \left(\frac{\alpha}{\alpha+\beta}\right)^2 \\ &= \frac{(\alpha+1)\alpha(\alpha+\beta) - \alpha^2(\alpha+\beta+1)}{(\alpha+\beta+1)(\alpha+\beta)^2} \qquad \text{(get common denominator).} \end{aligned}$$

It therefore suffices to show the numerator of this last expression equals $\alpha\beta$. Note that

$$\begin{aligned} (\alpha+1)\alpha(\alpha+\beta) - \alpha^2(\alpha+\beta+1) &= (\alpha^2+\alpha)(\alpha+\beta) - \alpha^3 - \alpha^2\beta - \alpha^2 \\ &= \alpha^3 + \alpha^2\beta + \alpha^2 + \alpha\beta - \alpha^3 - \alpha^2\beta - \alpha^2 = \alpha\beta. \ \square \end{aligned}$$

4.147. In this problem, we are not given the distribution of Y, the amount of cereal dispensed. All we know is Y is a random variable (measured in ounces) with mean μ and standard deviation σ. The phrase "the manufacturer wants Y to be within 1 ounce of μ at least 75% of the time" means

$$P(|Y-\mu| < 1) \geq 0.75.$$

Because we don't know the distribution of Y, the best we can do here is to use Tchebysheff's result; i.e.,

$$P(|Y-\mu|<k\sigma)\geq 1-\frac{1}{k^2}.$$

In this inequality, take $k=2$ to get

$$P(|Y-\mu|<2\sigma)\geq 1-\frac{1}{2^2}=0.75.$$

Therefore, for $P(|Y-\mu|<1)\geq 0.75$ to hold, the largest σ can be is $\sigma=0.5$ ounces.

4.157. I have noticed this type of problem appears on Exam P a lot−especially part (b). We are given that the lifetime of a component $Y\sim$ exponential($\beta=100$). Therefore, the pdf and cdf of Y are, respectively,

$$f_Y(y)=\begin{cases}\frac{1}{100}e^{-y/100}, & y>0\\ 0, & \text{otherwise}\end{cases}$$

and

$$F_Y(y)=\begin{cases}0, & y\leq 0\\ 1-e^{-y/100}, & y>0.\end{cases}$$

The component is replaced when it fails or at 200 hours, whichever comes first. Therefore, the time the component is in use is

$$X=g(Y)=\min\{Y,200\}=\begin{cases}Y, & Y<200\\ 200, & Y\geq 200.\end{cases}$$

For example, suppose the component fails at 100 hours. Then $y=100$ and $x=100$ too. Suppose the component would have failed at 300 hours. Here, $y=300$, but $x=200$ because the component is replaced at 200 hours.

(a) The cdf of X, the time the component is in use, is given by $F_X(x)=P(X\leq x)$. Clearly, when $x\leq 0$, the cdf is zero (i.e., time can only be positive). When $0<x<200$, the cdf is

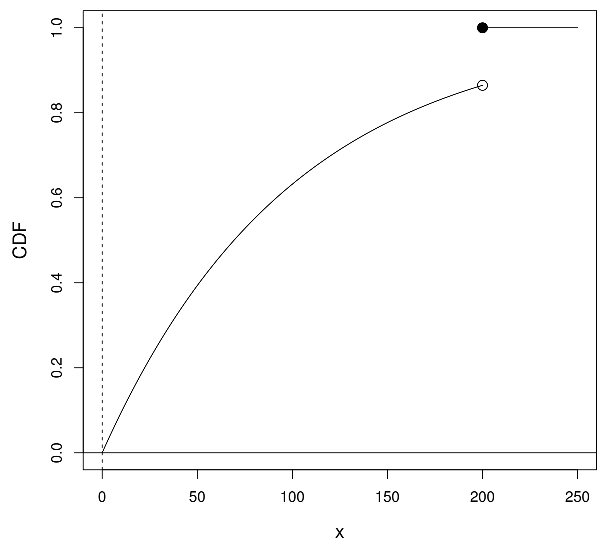
$$F_X(x)=P(X\leq x)=P(Y\leq x)=F_Y(x)=1-e^{-x/100}.$$

When $x\geq 200$, then the cdf is one because the largest X can be is 200. Therefore, the cdf of X is

$$F_X(x)=\begin{cases}0, & x\leq 0\\ 1-e^{-x/100}, & 0<x<200\\ 1, & x\geq 200.\end{cases}$$

A graph of $F_X(x)$ appears on the next page; here is the R code I used to produce it:

```
# Plot CDF
y = seq(0,200,0.1)
cdf = pexp(y,1/100)
plot(y,cdf,type="l",xlab="x",ylab="CDF",xlim=c(0,250),ylim=c(0,1),cex.lab=1.25)
abline(h=0)
```




```
abline(v=0,lty=2)
lines(c(200,250),c(1,1),lty=1)
points(x=200,y=1,pch=19,cex=1.5)
points(x=200,y=1-exp(-2),pch=1,cex=1.5)
```

(b) To find $E(X)$, first remember that X really is a function of Y; i.e.,

$$X = g(Y) = \min\{Y, 200\} = \begin{cases} Y, & Y < 200 \\ 200, & Y \geq 200. \end{cases}$$

Therefore, we want to calculate $E(X) = E[g(Y)]$. Using the definition of mathematical expectation, we have

$$E[g(Y)] = \int_{\mathbb{R}} g(y) f_Y(y) dy = \int_0^\infty \min\{y, 200\} \times \frac{1}{100} e^{-y/100} dy.$$

Now, when we are integrating over the region from 0 to 200, then the function $\min\{y, 200\} = y$. When we are integrating over the region from 200 to ∞, then $\min\{y, 200\} = 200$. Therefore, the last integral can be written as

$$\begin{aligned} E[g(Y)] &= \int_0^{200} y \times \frac{1}{100} e^{-y/100} dy \; + \; \int_{200}^\infty 200 \times \frac{1}{100} e^{-y/100} dy \\ &= \frac{1}{100} \underbrace{\int_0^{200} y e^{-y/100} dy}_{\text{integral 1}} \; + \; 2 \underbrace{\int_{200}^\infty e^{-y/100} dy}_{\text{integral 2}}. \end{aligned}$$

You can do the first integral using integration by parts:

$$\begin{aligned} u &= y & du &= dy \\ dv &= e^{-y/100} & v &= -100 e^{-y/100}. \end{aligned}$$

Therefore, the first integral is

$$\begin{aligned} -100ye^{-y/100}\Big|_{y=0}^{200} + 100\int_0^{200} e^{-y/100}dy &= -100\left(200e^{-2} - 0\right) + 100\left(-100e^{-y/100}\Big|_0^{200}\right) \\ &= -20000e^{-2} - 10000(e^{-2} - 1) \\ &= 10000 - 30000e^{-2} \approx 5939.942. \end{aligned}$$

The second integral is

$$-100e^{-y/100}\Big|_{200}^{\infty} = -100(0 - e^{-2}) = 100e^{-2} \approx 13.534.$$

Therefore,

$$E(X) = E[g(Y)] \approx \left(\frac{1}{100}\right) 5939.942 + 2(13.534) \approx 86.47 \text{ hours.}$$

4.182. A random variable Y is said to have a **lognormal distribution** with parameters μ and σ^2 if the pdf of Y is

$$f_Y(y) = \begin{cases} \dfrac{1}{\sqrt{2\pi}y\sigma} e^{-\frac{1}{2}\left(\frac{\ln y - \mu}{\sigma}\right)^2}, & y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

A lognormal random variable arises in the following way:

$$X \sim \mathcal{N}(\mu, \sigma^2) \implies Y = e^X \sim \text{lognormal}(\mu, \sigma^2),$$

or, equivalently,

$$Y \sim \text{lognormal}(\mu, \sigma^2) \implies X = \ln Y \sim \mathcal{N}(\mu, \sigma^2).$$

We will prove this result in Chapter 6. Although μ and σ^2 denote the mean and variance, respectively, in the normal distribution (for X), they are not the mean and variance for Y; see the formulas for $E(Y)$ and $V(Y)$ in Exercise 4.183.

Because of the relationship above, calculating probabilities for the lognormal distribution can be done using the normal distribution after transforming. Suppose $Y \sim \text{lognormal}(\mu = 4, \sigma^2 = 1)$. For part (a),

$$P(Y \le 4) = P(\ln Y \le \ln 4) = P(X \le \ln 4),$$

where $X \sim \mathcal{N}(\mu = 4, \sigma^2 = 1)$. This can be calculated in R using the `pnorm` function:

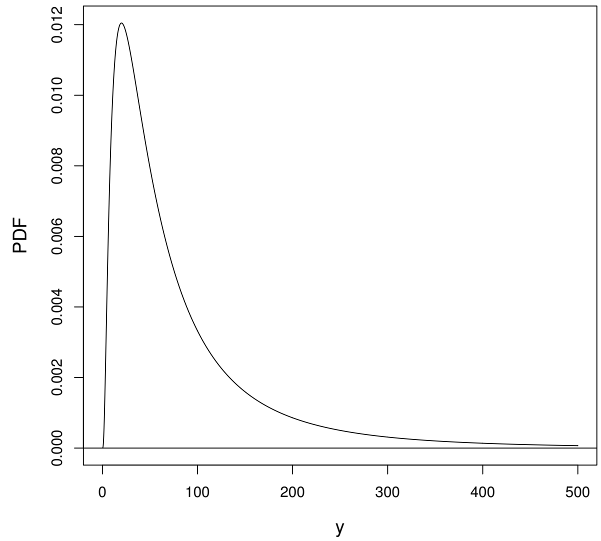
```
> pnorm(log(4),4,1)
[1] 0.004478308
```

For part (b),

$$P(Y > 8) = P(\ln Y > \ln 8) = P(X > \ln 8) = 1 - P(X \le \ln 8)$$

is calculated as

```
> 1-pnorm(log(8),4,1)
[1] 0.9726063
```



I used R to graph the pdf of $Y \sim \text{lognormal}(\mu = 4, \sigma^2 = 1)$; see the following code:

```
# Plot PDF
y = seq(0,500,0.1)
pdf = dlnorm(y,4,1)
plot(y,pdf,type="l",xlab="y",ylab="PDF",ylim=c(0,max(pdf)),cex.lab=1.25)
abline(h=0)
```

Interestingly, R can calculate lognormal probabilities using the `plnorm` function; note that

```
> plnorm(4,4,1)
[1] 0.004478308
> 1-plnorm(8,4,1)
[1] 0.9726063
```

Therefore, there really isn't a need to transform first, which is what the authors wanted you to do. R can calculate lognormal probabilities directly.

4.184. We first have to remember the definition of absolute value:

$$|y| = \begin{cases} y, & y \geq 0 \\ -y, & y < 0. \end{cases}$$

Therefore, when we are integrating over $[0, \infty)$, we use $|y| = y$. When we are integrating over $(-\infty, 0)$, we use $|y| = -y$. The mgf of Y is

$$\begin{aligned} m_Y(t) = E(e^{tY}) = \int_{-\infty}^{\infty} e^{ty} \times \frac{1}{2} e^{-|y|} dy &= \int_{-\infty}^{0} \frac{e^{ty}}{2} e^{y} dy + \int_{0}^{\infty} \frac{e^{ty}}{2} e^{-y} dy \\ &= \frac{1}{2} \underbrace{\int_{-\infty}^{0} e^{y(1+t)} dy}_{\text{integral 1}} + \frac{1}{2} \underbrace{\int_{0}^{\infty} e^{-y(1-t)} dy}_{\text{integral 2}}. \end{aligned}$$

The first integral is

$$\int_{-\infty}^{0} e^{y(1+t)}dy = \frac{1}{1+t}e^{y(1+t)}\bigg|_{-\infty}^{0} = \frac{1}{1+t}\left[1 - \lim_{y\to-\infty} e^{y(1+t)}\right] \stackrel{t>-1}{=} \frac{1}{1+t}(1-0) = \frac{1}{1+t}.$$

Note that

$$\lim_{y\to-\infty} e^{y(1+t)} = 0$$

only if $1+t>0 \iff t>-1$. If $1+t<0$, then this limit DNE. The second integral is

$$\int_{0}^{\infty} e^{-y(1-t)}dy = -\frac{1}{1-t}e^{-y(1-t)}\bigg|_{0}^{\infty} = -\frac{1}{1-t}\left[\lim_{y\to\infty} e^{-y(1-t)} - 1\right] \stackrel{t<1}{=} -\frac{1}{1-t}(0-1) = \frac{1}{1-t}.$$

Note that

$$\lim_{y\to\infty} e^{-y(1-t)} = 0$$

only if $1-t>0 \iff t<1$. If $1-t<0$, then this limit DNE.

Therefore, to ensure that both integrals converge, we need $-1<t<1$ (this includes an open neighborhood about zero). For these values of t, we have

$$m_Y(t) = \frac{1}{2}\left(\frac{1}{1+t}\right) + \frac{1}{2}\left(\frac{1}{1-t}\right) = \frac{1}{2}\left[\frac{1-t+1+t}{(1+t)(1-t)}\right] = \frac{1}{1-t^2}.$$

To find $E(Y)$, let's calculate the first derivative:

$$\frac{d}{dt}m_Y(t) = -1(1-t^2)^{-2}\times(-2t) = \frac{2t}{1-t^2}.$$

Therefore,

$$E(Y) = \frac{d}{dt}m_Y(t)\bigg|_{t=0} = \frac{2(0)}{1-(0)^2} = 0.$$

4.186. A random variable Y is said to have a **Weibull distribution** with parameters m and α if the pdf of Y is

$$f_Y(y) = \begin{cases} \frac{m}{\alpha}y^{m-1}e^{-y^m/\alpha}, & y>0 \\ 0, & \text{otherwise.} \end{cases}$$

Note that when $m=1$, this pdf reduces to an exponential pdf (with mean α). Interesting! Therefore, we can think of the Weibull(m,α) distribution as a generalization of the exponential.

When $m=2$, the pdf of Y is

$$f_Y(y) = \begin{cases} \frac{2y}{\alpha}e^{-y^2/\alpha}, & y>0 \\ 0, & \text{otherwise.} \end{cases}$$

The mean of Y is

$$E(Y) = \int_{\mathbb{R}} yf_Y(y)dy = \int_0^\infty \frac{2y^2}{\alpha}e^{-y^2/\alpha}dy.$$

In this last integral, let

$$u = y^2 \implies du = 2ydy.$$

Therefore, the last integral becomes

$$\int_0^\infty \frac{2y^2}{\alpha} e^{-u/\alpha} \frac{du}{2y} = \int_0^\infty \frac{\sqrt{u}}{\alpha} e^{-u/\alpha} du = \frac{1}{\alpha}\int_0^\infty u^{\frac{3}{2}-1} e^{-u/\alpha} du = \frac{1}{\alpha} \times \Gamma\left(\frac{3}{2}\right)\alpha^{3/2}$$
$$= \sqrt{\alpha}\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi\alpha}}{2}.$$

Recall that $\Gamma(1/2) = \sqrt{\pi}$, which we will prove coming up! Therefore,

$$E(Y) = \frac{\sqrt{\pi\alpha}}{2}.$$

To find $V(Y)$, we will first find $E(Y^2)$ and use the variance computing formula. Note that

$$E(Y^2) = \int_{\mathbb{R}} y^2 f_Y(y) dy = \int_0^\infty \frac{2y^3}{\alpha} e^{-y^2/\alpha} dy.$$

In this last integral, let

$$u = y^2 \implies du = 2ydy.$$

Therefore, the last integral becomes

$$\int_0^\infty \frac{2y^3}{\alpha} e^{-u/\alpha} \frac{du}{2y} = \int_0^\infty \frac{u}{\alpha} e^{-u/\alpha} du = E(U),$$

where $U \sim$ exponential(α). To see why this is true, note that $(1/\alpha)e^{-u/\alpha}$ is the exponential pdf with mean α, and we are integrating $u \times (1/\alpha)e^{-u/\alpha}$ over $(0, \infty)$. Therefore, $E(Y^2) = E(U) = \alpha$. Using the variance computing formula, we have

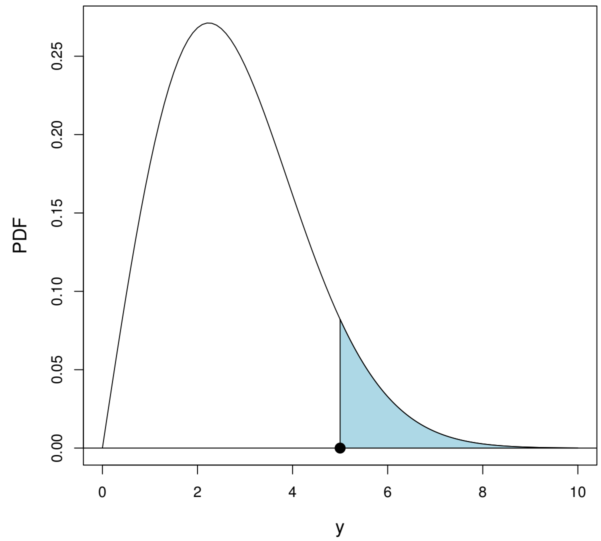
$$V(Y) = E(Y^2) - [E(Y)]^2 = \alpha - \left(\frac{\sqrt{\pi\alpha}}{2}\right)^2 = \alpha - \frac{\pi\alpha}{4} = \alpha\left(1 - \frac{\pi}{4}\right).$$

4.187. See Exercise 4.186. When $m = 2$ and $\alpha = 10$, the pdf of Y is

$$f_Y(y) = \begin{cases} \frac{2y}{10} e^{-y^2/10}, & y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

I used the following code to graph the pdf of Y; see next page.

```
# Plot PDF
y = seq(0,10,0.1)
pdf = dweibull(y,shape=2,scale=10^(1/2))
plot(y,pdf,type="l",xlab="y",ylab="PDF",ylim=c(0,max(pdf)),cex.lab=1.25)
abline(h=0)
# Shade in P(Y>5) in part (a)
x = seq(5,10,0.001)
y = dweibull(x,shape=2,scale=10^(1/2))
polygon(c(5,x,10),c(0,y,0),col="lightblue")
points(x=5,y=0,pch=19,cex=1.5)
```



(a) "Exceeds 5000 hours" means "$Y > 5$." We can find $P(Y > 5)$ by integrating the pdf; note that

$$P(Y > 5) = \int_5^\infty \frac{2y}{10} e^{-y^2/10} dy.$$

In this integral, let

$$u = y^2 \implies du = 2ydy.$$

Therefore,

$$\begin{aligned} P(Y > 5) = \int_{25}^\infty \frac{2y}{10} e^{-u/10} \ \frac{du}{2y} &= \int_{25}^\infty \frac{1}{10} e^{-u/10} du \\ &= \frac{1}{10}\left(-10e^{-u/10}\Big|_{25}^{\infty}\right) \\ &= e^{-u/10}\Big|_{\infty}^{25} = e^{-25/10} - \underbrace{\lim_{u\to\infty} e^{-u/10}}_{=0} = e^{-2.5} \approx 0.082. \end{aligned}$$

(b) Here we use the binomial distribution. Let

$$X = \text{number of resistors that fail } \textbf{before} \text{ 5000 hours of use.}$$

Assuming the resistors are independent, each with the same probability of failing before 5000 hours of use (i.e., $1 - 0.082 = 0.918$), we have $X \sim b(n = 3, p = 0.918)$. The probability exactly one fails before 5000 hours is

$$P(X = 1) = \binom{3}{1}(0.918)^1(1 - 0.918)^2 \approx 0.019.$$

In R,

```
> dbinom(1,3,0.918)
[1] 0.0185179
```

4.196. The hint is helpful. Note that

$$y = \frac{x^2}{2} \iff x = \sqrt{2y} \implies dx = \frac{1}{2}(2y)^{-1/2}(2)dy = \frac{dy}{\sqrt{2y}}.$$

Therefore,

$$\begin{aligned}
\Gamma(1/2) = \int_{y=0}^{\infty} y^{-1/2}e^{-y}dy &= \int_{x=0}^{\infty} \frac{1}{\sqrt{y}}e^{-x^2/2}\sqrt{2y}dx \\
&= \sqrt{2}\int_{x=0}^{\infty} e^{-x^2/2}dx \\
&= \sqrt{2}\sqrt{2\pi}\underbrace{\int_{x=0}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx}_{=\,1/2} = \frac{2\sqrt{\pi}}{2} = \sqrt{\pi}.
\end{aligned}$$

Note that

$$\int_{x=0}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx$$

is the integral of the standard normal pdf over $(0, \infty)$. We know

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx = 1.$$

Because the standard normal pdf is symmetric about $x = 0$, the median is zero; i.e., 1/2 the area is to the left of zero and 1/2 area is to the right.