4.128. Let $Y$ denote the weekly repair cost. Note that the nonzero part of the pdf $f_Y(y) = 3(1 - y)^2$ is what results in the beta($\alpha, \beta$) family when $\alpha = 1$ and $\beta = 3$; observe

\[
\frac{\Gamma(1+3)}{\Gamma(1)\Gamma(3)} y^{1-1}(1-y)^{3-1} = 3(1-y)^2.
\]

Therefore $Y \sim \text{beta}(1, 3)$. We want to find $\phi_{0.9}$, the $p = 0.9$ quantile of this distribution. Note that $\phi_{0.9}$ solves

\[
P(Y \leq \phi_{0.9}) = 0.9 \iff P(Y > \phi_{0.9}) = 0.1;
\]

i.e., the “cost will exceed...only 10% of the time.” We can find $\phi_{0.9}$ by solving

\[
0.9 = \int_0^{\phi_{0.9}} 3(1-y)^2 \, dy = -\int_1^{1-\phi_{0.9}} 3u^2 \, du \quad (u = 1-y)
\]

\[
= -u^3 \bigg|_1^{1-\phi_{0.9}} = -(1 - \phi_{0.9})^3 - 1 
\]

\[
\Rightarrow (1 - \phi_{0.9})^3 = 0.1 \Rightarrow \phi_{0.9} = 1 - (0.1)^{1/3} \approx 0.536.
\]

We could check our work in R using the `qbeta` function:

\[
> \text{qbeta}(0.9, 1, 3) \# p=0.9 \text{ quantile}
\]

\[
[1] 0.5358411
\]

Therefore, we would set the weekly budget at approximately 53.6 dollars; this would lead to exceeding the budget only about 10 percent of the time.

4.129. Let $Y$ denote the proportion of time the machine is down; $Y \sim \text{beta}(1, 2)$. Let

\[
C = 10 + 20Y + 4Y^2
\]

denote the cost of machine down time. The mean of $C$ is

\[
E(C) = E(10 + 20Y + 4Y^2) = 10 + 20E(Y) + 4E(Y^2).
\]

The beta mgf is not in a friendly form, so let’s not use it. We derived formulas for the mean and variance of a beta distribution; i.e.,

\[
E(Y) = \frac{1}{1+2} = \frac{1}{3} \quad \text{and} \quad V(Y) = \frac{1(2)}{(1+2)^2(1+2+1)} = \frac{2}{36} = \frac{1}{18}.
\]

Therefore, the second moment of $Y$ is

\[
E(Y^2) = V(Y) + [E(Y)]^2 = \frac{1}{18} + \left(\frac{1}{3}\right)^2 = \frac{1}{6}.
\]

Finally,

\[
E(C) = 10 + 20E(Y) + 4E(Y^2) = 10 + 20 \left(\frac{1}{3}\right) + 4 \left(\frac{1}{6}\right) = \frac{52}{3} \approx 17.33.
\]

Therefore, the expected (mean) cost due to down time is approximately $1,733.
Note: We could have calculated \( E(C) = E(10 + 20Y + 4Y^2) \) from first principles by writing
\[
E(10 + 20Y + 4Y^2) = \int_0^1 (10 + 20y + 4y^2) \times 2(1 - y)dy
\]
and then doing this integral. We would get the same answer. In fact, in R,
\[
> \text{integrand} = \text{function(y) \{(10+20*y+4*y^2)*2*(1-y)\}}
\]
\[
> \text{integrate(integrand,0,1)}
\]
17.33333 with absolute error < 1.9e-13

Getting the variance \( V(C) \) is harder if you are doing things by hand. An easy way you could do this is to write
\[
V(C) = E[(C - \mu_C)^2] = E \left[ (10 + 20Y + 4Y^2 - \frac{52}{3})^2 \right]
\]
\[
= \int_0^1 \left( 10 + 20y + 4y^2 - \frac{52}{3} \right)^2 \times 2(1 - y)dy.
\]
This integral can be calculated numerically in R:
\[
> \text{integrand.2 = function(y) \{(10+20*y+4*y^2-(52/3))^2*2*(1-y)\}}
\]
\[
> \text{integrate(integrand.2,0,1)}
\]
29.95556 with absolute error < 3.3e-13

Otherwise, we can do it as follows (the long way). First write
\[
V(C) = E(C^2) - [E(C)]^2 = E(C^2) - \left( \frac{52}{3} \right)^2.
\]

Now, we have to get \( E(C^2) \), the second moment of \( C \). Note that
\[
C^2 = (10 + 20Y + 4Y^2)^2 = 100 + 400Y^2 + 16Y^4 + 400Y + 80Y^2 + 160Y^3
\]
\[
= 100 + 400Y + 480Y^2 + 160Y^3 + 16Y^4.
\]

Therefore,
\[
E(C^2) = 100 + 400E(Y) + 480E(Y^2) + 160E(Y^3) + 16E(Y^4).
\]

We already know \( E(Y) = \frac{1}{3} \) and \( E(Y^2) = \frac{1}{6} \). The third moment of \( Y \sim \text{beta}(1,2) \) is
\[
E(Y^3) = \int_0^1 y^3 \times 2(1 - y)dy = 2 \int_0^1 \frac{y^{1-1}(1-y)^{2-1}}{\Gamma(4)\Gamma(2)} dy = \frac{2\Gamma(4)\Gamma(2)}{\Gamma(6)} = \frac{1}{10}.
\]

Similarly,
\[
E(Y^4) = \int_0^1 y^4 \times 2(1 - y)dy = 2 \int_0^1 \frac{y^{5-1}(1-y)^{2-1}}{\Gamma(5)\Gamma(2)} dy = \frac{2\Gamma(5)\Gamma(2)}{\Gamma(7)} = \frac{48}{720}.
\]

Therefore,
\[
E(C^2) = 100 + 400 \left( \frac{1}{3} \right) + 480 \left( \frac{1}{6} \right) + 160 \left( \frac{1}{10} \right) + 16 \left( \frac{48}{720} \right) = 330.4.
\]

Finally,
\[
V(C) = E(C^2) - [E(C)]^2 = 330.4 - \left( \frac{52}{3} \right)^2 \approx 29.96 \text{ (100s dollars)}^2.
\]
4.130. Here we are being asked to show

\[ V(Y) = \frac{\alpha \beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}, \]

where \( Y \sim \text{beta}(\alpha, \beta) \). In class, we derived

\[ E(Y) = \frac{\alpha}{\alpha + \beta}. \]

Recall that

\[ V(Y) = E(Y^2) - [E(Y)]^2 = E(Y^2) - \left( \frac{\alpha}{\alpha + \beta} \right)^2. \]

Therefore, all we have to do is to derive the second moment \( E(Y^2) \) and then do some algebra. Note that

\[ E(Y^2) = \int_0^1 y^2 \times \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} y^{\alpha-1}(1-y)^{\beta-1} dy = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 y^{\alpha+2-1}(1-y)^{\beta-1} dy \]

\[ = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha + 2)\Gamma(\beta)}{\Gamma(\alpha + \beta + 2)}. \]

Now use the recursive property of the gamma function to write

\[ \Gamma(\alpha + 2) = (\alpha + 1)\Gamma(\alpha + 1) = (\alpha + 1)\alpha\Gamma(\alpha) \]

and

\[ \Gamma(\alpha + \beta + 2) = (\alpha + \beta + 1)\Gamma(\alpha + \beta + 1) = (\alpha + \beta + 1)(\alpha + \beta)\Gamma(\alpha + \beta). \]

Therefore, \( E(Y^2) \) above becomes

\[ E(Y^2) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{(\alpha + 1)\alpha\Gamma(\alpha)\Gamma(\beta)}{(\alpha + \beta + 1)(\alpha + \beta)\Gamma(\alpha + \beta)} = \frac{(\alpha + 1)\alpha}{(\alpha + \beta + 1)(\alpha + \beta)}. \]

Therefore,

\[ V(Y) = E(Y^2) - [E(Y)]^2 = \frac{(\alpha + 1)\alpha}{(\alpha + \beta + 1)(\alpha + \beta)} - \left( \frac{\alpha}{\alpha + \beta} \right)^2 \]

\[ = \frac{(\alpha + 1)\alpha(\alpha + \beta) - \alpha^2(\alpha + \beta + 1)}{(\alpha + \beta + 1)(\alpha + \beta)^2} \text{ (get common denominator).} \]

It therefore suffices to show the numerator of this last expression equals \( \alpha \beta \). Note that

\[ (\alpha + 1)\alpha(\alpha + \beta) - \alpha^2(\alpha + \beta + 1) = (\alpha^2 + \alpha)(\alpha + \beta) - \alpha^3 - \alpha^2 \beta - \alpha^2 \]

\[ = \alpha^3 + \alpha^2 \beta + \alpha^2 + \alpha \beta - \alpha^3 - \alpha^2 \beta - \alpha^2 = \alpha \beta. \]

\[ \square \]

4.147. In this problem, we are not given the distribution of \( Y \), the amount of cereal dispensed. All we know is \( Y \) is a random variable (measured in ounces) with mean \( \mu \) and standard deviation \( \sigma \). The phrase “the manufacturer wants \( Y \) to be within 1 ounce of \( \mu \) at least 75% of the time” means

\[ P(|Y - \mu| < 1) \geq 0.75. \]
Because we don’t know the distribution of \( Y \), the best we can do here is to use Tchebysheff’s result; i.e.,

\[
P(|Y - \mu| < k\sigma) \geq 1 - \frac{1}{k^2}.
\]

In this inequality, take \( k = 2 \) to get

\[
P(|Y - \mu| < 2\sigma) \geq 1 - \frac{1}{2^2} = 0.75.
\]

Therefore, for \( P(|Y - \mu| < 1) \geq 0.75 \) to hold, the largest \( \sigma \) can be is \( \sigma = 0.5 \) ounces.

**4.157.** I have noticed this type of problem appears on Exam P a lot—especially part (b). We are given that the lifetime of a component \( Y \sim \text{exponential}(\beta = 100) \). Therefore, the pdf and cdf of \( Y \) are, respectively,

\[
f_Y(y) = \begin{cases} 
\frac{1}{100} e^{-y/100}, & y > 0 \\
0, & \text{otherwise}
\end{cases}
\]

and

\[
F_Y(y) = \begin{cases} 
0, & y \leq 0 \\
1 - e^{-y/100}, & y > 0.
\end{cases}
\]

The component is replaced when it fails or at 200 hours, whichever comes first. Therefore, the time the component is in use is

\[X = g(Y) = \min\{Y, 200\} = \begin{cases} 
Y, & Y < 200 \\
200, & Y \geq 200.
\end{cases}\]

For example, suppose the component fails at 100 hours. Then \( y = 100 \) and \( x = 100 \) too. Suppose the component would have failed at 300 hours. Here, \( y = 300 \), but \( x = 200 \) because the component is replaced at 200 hours.

(a) The cdf of \( X \), the time the component is in use, is given by \( F_X(x) = P(X \leq x) \). Clearly, when \( x \leq 0 \), the cdf is zero (i.e., time can only be positive). When \( 0 < x < 200 \), the cdf is

\[
F_X(x) = P(X \leq x) = P(Y \leq x) = F_Y(x) = 1 - e^{-x/100}.
\]

When \( x \geq 200 \), then the cdf is one because the largest \( X \) can be is 200. Therefore, the cdf of \( X \) is

\[
F_X(x) = \begin{cases} 
0, & x \leq 0 \\
1 - \frac{1}{100} e^{-x}, & 0 < x < 200 \\
1, & x \geq 200.
\end{cases}
\]

A graph of \( F_X(x) \) appears on the next page; here is the R code I used to produce it:

```r
# Plot CDF
y = seq(0,200,0.1)
cdf = pexp(y,1/100)
plot(y,cdf,type="l",xlab="x",ylab="CDF",xlim=c(0,250),ylim=c(0,1),cex.lab=1.25)
abline(h=0)
```
(b) To find \( E(X) \), first remember that \( X \) really is a function of \( Y \); i.e.,

\[
X = g(Y) = \min\{Y, 200\} = \begin{cases} Y, & Y < 200 \\ 200, & Y \geq 200. \end{cases}
\]

Therefore, we want to calculate \( E(X) = E[g(Y)] \). Using the definition of mathematical expectation, we have

\[
E[g(Y)] = \int_{\mathbb{R}} g(y)f_Y(y)\,dy = \int_{0}^{\infty} \min\{y, 200\} \times \frac{1}{100} e^{-y/100} \,dy.
\]

Now, when we are integrating over the region from 0 to 200, then the function \( \min\{y, 200\} = y \). When we are integrating over the region from 200 to \( \infty \), then \( \min\{y, 200\} = 200 \). Therefore, the last integral can be written as

\[
E[g(Y)] = \int_{0}^{200} y \times \frac{1}{100} e^{-y/100} \,dy + \int_{200}^{\infty} 200 \times \frac{1}{100} e^{-y/100} \,dy
\]

\[
= \frac{1}{100} \int_{0}^{200} ye^{-y/100} \,dy + 2 \int_{200}^{\infty} e^{-y/100} \,dy.
\]

You can do the first integral using integration by parts:

\[
\begin{align*}
u &= y & dv &= e^{-y/100} \\
du &= dy & v &= -100e^{-y/100}.
\end{align*}
\]
Therefore, the first integral is
\[
-100ye^{-y/100}\bigg|_{y=0}^{200} + 100 \int_0^{200} e^{-y/100} dy = -100 \left(200e^{-2} - 0\right) + 100 \left(-100e^{-y/100}\bigg|_0^{200}\right) \\
= -20000e^{-2} - 10000(e^{-2} - 1) \\
= 10000 - 30000e^{-2} \approx 5939.942.
\]

The second integral is
\[
-100e^{-y/100}\bigg|_0^\infty = -100(0 - e^{-2}) = 100e^{-2} \approx 13.534.
\]

Therefore,
\[
E(X) = E[g(Y)] \approx \left(\frac{1}{100}\right) 5939.942 + 2(13.534) \approx 86.47 \text{ hours}.
\]

4.182. A random variable \(Y\) is said to have a lognormal distribution with parameters \(\mu\) and \(\sigma^2\) if the pdf of \(Y\) is
\[
f_Y(y) = \begin{cases} 
\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}\left(\ln y - \mu\right)^2}, & y > 0 \\
0, & \text{otherwise.}
\end{cases}
\]

A lognormal random variable arises in the following way:
\[
X \sim \mathcal{N}(\mu, \sigma^2) \implies Y = e^X \sim \text{lognormal}(\mu, \sigma^2),
\]
or, equivalently,
\[
Y \sim \text{lognormal}(\mu, \sigma^2) \implies X = \ln Y \sim \mathcal{N}(\mu, \sigma^2).
\]

We will prove this result in Chapter 6. Although \(\mu\) and \(\sigma^2\) denote the mean and variance, respectively, in the normal distribution (for \(X\)), they are not the mean and variance for \(Y\); see the formulas for \(E(Y)\) and \(V(Y)\) in Exercise 4.183.

Because of the relationship above, calculating probabilities for the lognormal distribution can be done using the normal distribution after transforming. Suppose \(Y \sim \text{lognormal}(\mu = 4, \sigma^2 = 1)\).

For part (a),
\[
P(Y \leq 4) = P(\ln Y \leq \ln 4) = P(X \leq \ln 4),
\]
where \(X \sim \mathcal{N}(\mu = 4, \sigma^2 = 1)\). This can be calculated in R using the `pnorm` function:

\[
> \text{pnorm(log(4),4,1)}
\]
\[
[1] 0.004478308
\]

For part (b),
\[
P(Y > 8) = P(\ln Y > \ln 8) = P(X > \ln 8) = 1 - P(X \leq \ln 8)
\]
is calculated as
\[
> 1 - \text{pnorm(\text{log(8)},4,1)}
\]
\[
[1] 0.9726063
\]
I used R to graph the pdf of \( Y \sim \text{lognormal}(\mu = 4, \sigma^2 = 1) \); see the following code:

```r
# Plot PDF
y = seq(0,500,0.1)
pdf = dlnorm(y,4,1)
plot(y,pdf,type="l",xlab="y",ylab="PDF",ylim=c(0,max(pdf)),cex.lab=1.25)
abline(h=0)
```

Interestingly, R can calculate lognormal probabilities using the `plnorm` function; note that

```r
> plnorm(4,4,1)
[1] 0.004478308
> 1-plnorm(8,4,1)
[1] 0.9726063
```

Therefore, there really isn’t a need to transform first, which is what the authors wanted you to do. R can calculate lognormal probabilities directly.

4.184. We first have to remember the definition of absolute value:

\[
|y| = \begin{cases} 
y, & y \geq 0 

\end{cases}

\begin{cases} 
-y, & y < 0.
\end{cases}
\]

Therefore, when we are integrating over \([0, \infty)\), we use \(|y| = y\). When we are integrating over \((-\infty, 0)\), we use \(|y| = -y\). The mgf of \( Y \) is

\[
m_Y(t) = E(e^{tY}) = \int_{-\infty}^{\infty} e^{ty} \times \frac{1}{2} e^{-|y|} dy = \int_{-\infty}^{0} \frac{e^{ty}}{2} e^{y} dy + \int_{0}^{\infty} \frac{e^{ty}}{2} e^{-y} dy
\]

\[
= \frac{1}{2}\int_{-\infty}^{0} e^{y(1+t)} dy + \frac{1}{2}\int_{0}^{\infty} e^{-y(1-t)} dy.
\]

```
The first integral is
\[ \int_{-\infty}^{0} e^{y(1+t)} \, dy = \frac{1}{1+t} \left. e^{y(1+t)} \right|_{-\infty}^{0} = \frac{1}{1+t} \left[ 1 - \lim_{y \to -\infty} e^{y(1+t)} \right] \]
\[ \overset{t>-1}{=} \frac{1}{1+t} (1-0) = \frac{1}{1+t}. \]

Note that
\[ \lim_{y \to -\infty} e^{y(1+t)} = 0 \]
only if \( 1 + t > 0 \iff t > -1. \) If \( 1 + t < 0, \) then this limit DNE. The second integral is
\[ \int_{0}^{\infty} e^{-y(1-t)} \, dy = -\frac{1}{1-t} e^{-y(1-t)} \bigg|_{0}^{\infty} = -\frac{1}{1-t} \left[ \lim_{y \to \infty} e^{-y(1-t)} - 1 \right] \overset{t<1}{=} -\frac{1}{1-t} (0 - 1) = \frac{1}{1-t}. \]

Note that
\[ \lim_{y \to \infty} e^{-y(1-t)} = 0 \]
only if \( 1 - t > 0 \iff t < 1. \) If \( 1 - t < 0, \) then this limit DNE.

Therefore, to ensure that both integrals converge, we need \(-1 < t < 1\) (this includes an open neighborhood about zero). For these values of \( t, \) we have
\[ m_Y(t) = \frac{1}{2} \left( \frac{1}{1+t} \right) + \frac{1}{2} \left( \frac{1}{1-t} \right) = \frac{1}{2} \left[ \frac{1-t + t}{(1+t)(1-t)} \right] = \frac{1}{1-t^2}. \]

To find \( E(Y), \) let’s calculate the first derivative:
\[ \frac{d}{dt} m_Y(t) = -1(1-t^2)^{-2} \times (-2t) = \frac{2t}{1-t^2}. \]

Therefore,
\[ E(Y) = \left. \frac{d}{dt} m_Y(t) \right|_{t=0} = \frac{2(0)}{1-(0)^2} = 0. \]

4.186. A random variable \( Y \) is said to have a **Weibull distribution** with parameters \( m \) and \( \alpha \) if the pdf of \( Y \) is
\[ f_Y(y) = \begin{cases} \frac{m}{\alpha} y^{m-1} e^{-y^m/\alpha}, & y > 0 \\ 0, & \text{otherwise.} \end{cases} \]

Note that when \( m = 1, \) this pdf reduces to an exponential pdf (with mean \( \alpha \)). Interesting!

Therefore, we can think of the Weibull\((m, \alpha)\) distribution as a generalization of the exponential.

When \( m = 2, \) the pdf of \( Y \) is
\[ f_Y(y) = \begin{cases} \frac{2y}{\alpha} e^{-y^2/\alpha}, & y > 0 \\ 0, & \text{otherwise.} \end{cases} \]

The mean of \( Y \) is
\[ E(Y) = \int_{\mathbb{R}} y f_Y(y) \, dy = \int_{0}^{\infty} \frac{2y^2}{\alpha} e^{-y^2/\alpha} \, dy. \]

In this last integral, let
\[ u = y^2 \iff du = 2y \, dy. \]
Therefore, the last integral becomes
\[
\int_{0}^{\infty} \frac{2y^2}{\alpha} e^{-u/\alpha} \frac{du}{2y} = \int_{0}^{\infty} \frac{\sqrt{u}}{\alpha} e^{-u/\alpha} du = \frac{1}{\alpha} \int_{0}^{\infty} u^{3/2-1} e^{-u/\alpha} du = \frac{1}{\alpha} \times \Gamma\left(\frac{3}{2}\right) \alpha^{3/2}
\]
\[
= \sqrt{\alpha} \left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi} \alpha}{2}.
\]
Recall that \(\Gamma(1/2) = \sqrt{\pi}\), which we will prove coming up! Therefore,
\[
E(Y) = \frac{\sqrt{\pi} \alpha}{2}.
\]
To find \(V(Y)\), we will first find \(E(Y^2)\) and use the variance computing formula. Note that
\[
E(Y^2) = \int_{\mathbb{R}} y^2 f_Y(y) dy = \int_{0}^{\infty} \frac{2y^3}{\alpha} e^{-y^2/\alpha} dy.
\]
In this last integral, let
\[
u = y^2 \implies du = 2ydy.
\]
Therefore, the last integral becomes
\[
\int_{0}^{\infty} \frac{2y^3}{\alpha} e^{-u/\alpha} \frac{du}{2y} = \int_{0}^{\infty} \frac{u}{\alpha} e^{-u/\alpha} du = E(U),
\]
where \(U \sim \text{exponential}(\alpha)\). To see why this is true, note that \((1/\alpha)e^{-u/\alpha}\) is the exponential pdf with mean \(\alpha\), and we are integrating \(u \times (1/\alpha)e^{-u/\alpha}\) over \((0, \infty)\). Therefore, \(E(Y^2) = E(U) = \alpha\).
Using the variance computing formula, we have
\[
V(Y) = E(Y^2) - [E(Y)]^2 = \alpha - \left(\frac{\sqrt{\pi} \alpha}{2}\right)^2 = \alpha - \frac{\pi \alpha^2}{4} = \alpha \left(1 - \frac{\pi}{4}\right).
\]

4.187. See Exercise 4.186. When \(m = 2\) and \(\alpha = 10\), the pdf of \(Y\) is
\[
f_Y(y) = \begin{cases} 
\frac{2y}{10} e^{-y^2/10}, & y > 0 \\
0, & \text{otherwise}.
\end{cases}
\]
I used the following code to graph the pdf of \(Y\); see next page.

```python
# Plot PDF
y = seq(0,10,0.1)
pdf = dweibull(y,shape=2,scale=10^(1/2))
plot(y,pdf,type="l",xlab="y",ylab="PDF",ylim=c(0,max(pdf)),cex.lab=1.25)
abline(h=0)
# Shade in P(Y>5) in part (a)
x = seq(5,10,0.001)
y = dweibull(x,shape=2,scale=10^(1/2))
polygon(c(5,x,10),c(0,y,0),col="lightblue")
points(x=5,y=0,pch=19,cex=1.5)
```
(a) "Exceeds 5000 hours" means "$Y > 5$." We can find $P(Y > 5)$ by integrating the pdf; note that

$$P(Y > 5) = \int_5^\infty \frac{2y}{10} e^{-y^2/10} \, dy.$$ 

In this integral, let

$$u = y^2 \quad \Rightarrow \quad du = 2y \, dy.$$ 

Therefore,

$$P(Y > 5) = \int_{25}^\infty \frac{2y}{10} e^{-u/10} \frac{du}{2y} = \int_{25}^\infty \frac{1}{10} e^{-u/10} \, du = \frac{1}{10} \left(-10e^{-u/10}\right)_{25}^{\infty} = e^{-u/10}\bigg|_{25}^{\infty} = e^{-25/10} - \lim_{u \to \infty} e^{-u/10} = e^{-2.5} \approx 0.082.$$ 

(b) Here we use the binomial distribution. Let

$$X = \text{number of resistors that fail before 5000 hours of use.}$$

Assuming the resistors are independent, each with the same probability of failing before 5000 hours of use (i.e., $1 - 0.082 = 0.918$), we have $X \sim b(n = 3, p = 0.918)$. The probability exactly one fails before 5000 hours is

$$P(X = 1) = \binom{3}{1} (0.918)^1 (1 - 0.918)^2 \approx 0.019.$$ 

In R,

> dbinom(1,3,0.918)
> [1] 0.0185179
4.196. The hint is helpful. Note that

\[ y = \frac{x^2}{2} \iff x = \sqrt{2y} \implies dx = \frac{1}{2}(2y)^{-1/2}(2)dy = \frac{dy}{\sqrt{2y}}. \]

Therefore,

\[
\Gamma(1/2) = \int_{y=0}^{\infty} y^{-1/2}e^{-y}dy = \int_{x=0}^{\infty} \frac{1}{\sqrt{y}}e^{-x^2/2}\sqrt{2y}dx \\
= \sqrt{2} \int_{x=0}^{\infty} e^{-x^2/2}dx \\
= \sqrt{2\sqrt{2\pi}} \int_{x=0}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx = \frac{2\sqrt{\pi}}{2} = \sqrt{\pi}.
\]

Note that

\[
\int_{x=0}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx
\]

is the integral of the standard normal pdf over \((0, \infty)\). We know

\[
\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx = 1.
\]

Because the standard normal pdf is symmetric about \(x = 0\), the median is zero; i.e., 1/2 the area is to the left of zero and 1/2 area is to the right.