1. (a) Let $Y$ denote the number of defective items (out of 30). Then $Y \sim b(n=30, p = 0.05)$. The probability there are at most two defective items is

$$P(Y \leq 2) = P(Y = 0) + P(Y = 1) + P(Y = 2)$$

$$= \binom{30}{0}(0.05)^0(0.95)^{30} + \binom{30}{1}(0.05)^1(0.95)^{29} + \binom{30}{2}(0.05)^2(0.95)^{28}$$

$$\approx 0.215 + 0.339 + 0.259 = 0.813.$$  

(b) In this part, let $Y$ denote the number of items observed to find the first defective item. We know $Y \sim \text{geom}(p = 0.05)$. From the complement rule,

$$P(Y > 20) = 1 - P(Y \leq 20).$$

Now,

$$P(Y \leq 20) = \sum_{y=1}^{20} (0.95)^{y-1}(0.05) = 0.05 \sum_{x=0}^{19} (0.95)^x.$$  

I let $x = y - 1$ in the last step. Note that $\sum_{x=0}^{19} (0.95)^x$ is a finite geometric sum with common ratio $r = 0.95$; i.e.,

$$\sum_{x=0}^{19} (0.95)^x = \frac{1 - (0.95)^{20}}{1 - 0.95}.$$  

Therefore,

$$P(Y \leq 20) = 0.05 \left[ \frac{1 - (0.95)^{20}}{1 - 0.95} \right] = 1 - (0.95)^{20} \approx 0.642.$$  

Finally,

$$P(Y > 20) = 1 - P(Y \leq 20) \approx 1 - 0.642 = 0.358.$$  

2. (a) Let $Y$ denote the number of claims filed in a given day. The phrase “three times as likely to file 2 claims as to file 4 claims” means

$$P(Y = 2) = 3P(Y = 4).$$

Using the Poisson pmf $p_Y(y) = P(Y = y)$, this means

$$\frac{\lambda^2 e^{-\lambda}}{2!} = 3 \times \frac{\lambda^4 e^{-\lambda}}{4!}.$$  

Now, solve this equation for $\lambda$. We have

$$\frac{\lambda^2}{2} = \frac{\lambda^4}{8} \Rightarrow 8\lambda^2 = 2\lambda^4 \Rightarrow 4 = \lambda^2 \Rightarrow \lambda = \pm 2.$$  

We know $\lambda > 0$, so $\lambda = 2$.  

(b) The mode is the most likely value of $Y$; i.e., the value of $y$ that maximizes $p_Y(y) =$
Let’s calculate the first few values of $P(Y = y)$ assuming $\lambda = 2$.

\[
P(Y = 0) = \frac{2^0 e^{-2}}{0!} = e^{-2} \approx 0.135
\]
\[
P(Y = 1) = \frac{2^1 e^{-2}}{1!} = 2e^{-2} \approx 0.271
\]
\[
P(Y = 2) = \frac{2^2 e^{-2}}{2!} = 2e^{-2} \approx 0.271
\]
\[
P(Y = 3) = \frac{2^3 e^{-2}}{3!} = \frac{8e^{-2}}{6} \approx 0.180
\]
\[
P(Y = 4) = \frac{2^4 e^{-2}}{4!} = \frac{16e^{-2}}{24} \approx 0.090
\]
\[
P(Y = 5) = \frac{2^5 e^{-2}}{5!} = \frac{32e^{-2}}{120} \approx 0.036
\]

Probabilities $P(Y = y)$, for $y = 6, 7, 8, \ldots$, continue to decrease. Therefore, the mode of $Y$ is $y = 1$ or $y = 2$; i.e., $Y$ has a double mode.

3. (a) We know $f_Y(y)$ integrates to 1 over its support so

\[
1 \seteq \int_{1}^{\infty} \frac{c}{y^3} dy = c \left( -\frac{1}{2} y^{-2} \right) \bigg|_{1}^{\infty} = \frac{c}{2} \left( 1 - \lim_{y \to \infty} \frac{1}{y^2} \right) = \frac{c}{2} \implies c = 2.
\]

(b) The cdf $F_Y(y) = P(Y \leq y)$ is defined for all $y \in \mathbb{R}$. Thus, we consider two cases:

\textbf{Case 1: } $y < 1$.

\[
F_Y(y) = P(Y \leq y) = \int_{-\infty}^{y} f_Y(t) dt = \int_{-\infty}^{y} 0 dt = 0.
\]

\textbf{Case 2: } $y \geq 1$.

\[
F_Y(y) = P(Y \leq y) = \int_{-\infty}^{y} f_Y(t) dt = \left[ \int_{-\infty}^{1} 0 dt + \int_{1}^{y} \frac{2}{t^3} dt \right] = 0 + \left[ -\frac{1}{2} t^{-2} \right]_{1}^{y} = 1 - \frac{1}{y^2}.
\]

Summarizing,

\[
F_Y(y) = \begin{cases} 
0, & y < 1 \\
1 - \frac{1}{y^2}, & y \geq 1.
\end{cases}
\]

(c) To find the 99th percentile $\phi_{0.99}$, we set the cdf of $Y$ equal to 0.99 and solve for $\phi_{0.99}$; i.e.,

\[
0.99 \seteq P(Y \leq \phi_{0.99}) = F_Y(\phi_{0.99}) = 1 - \frac{1}{\phi_{0.99}^2} \implies \frac{1}{\phi_{0.99}^2} = 0.01 \implies \phi_{0.99} = \pm 10.
\]

We know the solution must be $\geq 1$. Therefore, the 99th percentile is 10 micrometers.
4. (a) It’s easiest to use the mgf. Note that
\[
\frac{d}{dt} m_Z(t) = \frac{d}{dt} e^{t^2/2} = te^{t^2/2}.
\]
Therefore,
\[
E(Z) = \frac{d}{dt} m_Z(t) \bigg|_{t=0} = te^{t^2/2} \bigg|_{t=0} = 0 e^{(0)^2/2} = 0.
\]
(b) First, remember the definition of absolute value:
\[
|z| = \begin{cases} 
z, & z \geq 0 \-z, & z < 0. \end{cases}
\]
Note that
\[
E(|Z|) = \int_{-\infty}^{\infty} |z| \times \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = \int_{-\infty}^{\infty} h(z) dz,
\]
where the function
\[
h(z) = \frac{|z|}{\sqrt{2\pi}} e^{-z^2/2}.
\]
Following the hint, note that
\[
h(z) = \frac{|z|}{\sqrt{2\pi}} e^{-z^2/2} = \frac{|-z|}{\sqrt{2\pi}} e^{-(z)^2/2} = h(-z);
\]
i.e., the function \(h\) is symmetric about zero. Because \(h(z) \geq 0\) for all \(z \in \mathbb{R}\), we know \(\int_{-\infty}^{0} h(z) dz = \int_{0}^{\infty} h(z) dz\) and hence
\[
\int_{-\infty}^{\infty} h(z) dz = \int_{0}^{\infty} h(z) dz + \int_{0}^{\infty} h(z) dz = 2 \int_{0}^{\infty} h(z) dz.
\]
Therefore,
\[
E(|Z|) = 2 \int_{0}^{\infty} h(z) dz = 2 \int_{0}^{\infty} \frac{z}{\sqrt{2\pi}} e^{-z^2/2} dz \quad (z \geq 0).
\]
The antiderivative of \(ze^{-z^2/2}\) is \(-e^{-z^2/2}\), so
\[
E(|Z|) = \frac{2}{\sqrt{2\pi}} \left( e^{-z^2/2} \right|_{0}^{\infty} = \frac{2}{\sqrt{2\pi}} \left( 1 - \lim_{z \to \infty} e^{-z^2/2} \right) = \frac{2}{\sqrt{2\pi}} (1 - 0) = \sqrt{\frac{2}{\pi}}.
\]
5. This problem is very easy if you remember the exponential cdf; i.e.,
\[
F_Y(y) = \begin{cases} 
0, & y \leq 0 
1 - e^{-y/\beta}, & y > 0.
\end{cases}
\]
(a) With \(\beta = 4\),
\[
P(Y < 5) = F_Y(5) = 1 - e^{-5/4} \approx 0.713.
\]
Note: If you did not remember the exponential cdf, you could always integrate the pdf; i.e.,

\[ P(Y < 5) = \int_0^5 \frac{1}{4} e^{-y/4} \, dy = \frac{1}{4} \left( -4e^{-y/4} \right]_0^5 = 1 - e^{-5/4} \approx 0.713. \]

(b) The conditional probability

\[ P(Y > 5 | Y > 2) = \frac{P(Y > 5 \text{ and } Y > 2)}{P(Y > 2)} = \frac{P(Y > 5)}{P(Y > 2)} = \frac{1 - F_Y(5)}{1 - F_Y(2)} = \frac{1 - (1 - e^{-5/4})}{1 - (1 - e^{-2/4})} = e^{-3/4} \approx 0.472. \]

Also,

\[ P(Y > 3) = 1 - F_Y(3) = 1 - (1 - e^{-3/4}) = e^{-3/4} \approx 0.472. \]

Therefore, \( P(Y > 5 | Y > 2) = P(Y > 3) \). This is a consequence of the memoryless property of the exponential distribution.

6. (a) Look at the beta kernel \( y^7(1-y) \) and compare it with the general beta pdf:

\[ \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} y^{\alpha-1}(1-y)^{\beta-1}. \]

Therefore, \( \alpha = 8 \) and \( \beta = 2 \). Therefore,

\[ k = \frac{\Gamma(8 + 2)}{\Gamma(8)\Gamma(2)} = \frac{9!}{7!} = 9 \times 8 = 72. \]

(b) The median \( \phi_{0.5} \) solves

\[ \int_0^{\phi_{0.5}} 72y^7(1-y) \, dy = 0.5 \quad \text{or} \quad \int_{\phi_{0.5}}^1 72y^7(1-y) \, dy = 0.5. \]

(c) The expected cost is \( E(C) = E(12 + 2Y + 5Y^2) = 12 + 2E(Y) + 5E(Y^2) \). We know \( E(Y) = 8/(8+2) = 0.8 \). To find the second moment, we could note

\[ E(Y^2) = \int_0^1 y^2 \times 72y^7(1-y) \, dy = 72 \int_0^1 \frac{y^9(1-y)}{\text{beta}(10, 2) \text{ kernel}} \, dy = \frac{72\Gamma(10)\Gamma(2)}{\Gamma(12)} = \frac{72 \times 9!}{11!} = \frac{72}{110}. \]

Therefore,

\[ E(C) = E(12 + 2Y + 5Y^2) = 12 + 2(0.8) + 5 \left( \frac{72}{110} \right) \approx 16.87. \]

Note: You could also get \( E(Y^2) \) if you used \( V(Y) = E(Y^2) - [E(Y)]^2 \), but you would have to remember the formula for \( V(Y) \) for the beta distribution.