6.4. The amount of flour used is a random variable $Y \sim \text{exponential}(\beta = 4)$. The pdf of $Y$ is shown below (left). The cost is described in terms of a function of $Y$; i.e., $U = h(Y) = 3Y + 1$.

(a) Let’s use the cdf technique to find the distribution of $U$. First, note that $y > 0 \iff u = 3y + 1 > 1$. Therefore, the support of $U$ is $R_U = \{u : u > 1\}$. For $u > 1$, the cdf of $U$ is

$$F_U(u) = P(U \leq u) = P(3Y + 1 \leq u) = P\left(Y \leq \frac{u-1}{3}\right) = F_Y\left(\frac{u-1}{3}\right).$$

Recall the cdf of $Y \sim \text{exponential}(\beta = 4)$ is

$$F_Y(y) = \begin{cases} 
0, & y \leq 0 \\
1 - e^{-y/4}, & y > 0.
\end{cases}$$

Therefore, for $y > 0 \iff u > 1$, the cdf of $U = h(Y) = 3Y + 1$ is

$$F_U(u) = F_Y\left(\frac{u-1}{3}\right) = 1 - e^{-\left(\frac{u+1}{4}\right)/4} = 1 - e^{-(u-1)/12}.$$

For $u > 1$, the pdf of $U$ is

$$f_U(u) = \frac{d}{du} F_U(u) = \frac{d}{du} \left\{1 - e^{-(u-1)/12}\right\} = \frac{1}{12} e^{-(u-1)/12}.$$

Summarizing,

$$f_U(u) = \begin{cases} 
\frac{1}{12} e^{-(u-1)/12}, & u > 1 \\
0, & \text{otherwise.}
\end{cases}$$

The pdf of $U$ is shown below (right). Note that $f_U(u)$ is an exponential(12) pdf but with a horizontal shift of 1 unit to the right; i.e., a “shifted-exponential distribution.”

(b) Using the pdf from part (a), the mean of $U$ is

$$E(U) = \int_{\mathbb{R}} u f_U(u) du = \int_1^{\infty} \frac{u}{12} e^{-(u-1)/12} du.$$
In the last integral, let $v = u - 1$ so that $dv = du$. Therefore,

$$E(U) = \int_{0}^{\infty} (v + 1) \frac{1}{12} e^{-v/12} dv = E(V + 1),$$

where $V \sim \text{exponential}(12)$. We have $E(U) = E(V + 1) = E(V) + 1 = 12 + 1 = 13$. Of course, we would get the same answer by using the Law of the Unconscious Statistician; note that

$$E(U) = E(3Y + 1) = 3E(Y) + 1 = 3(4) + 1 = 13.$$

**6.5.** The waiting time is a random variable $Y \sim U(1, 5)$. The pdf of $Y$ is shown above (left). The cost is described in terms of a function of $Y$; i.e., $U = h(Y) = 2Y^2 + 3$.

Let’s use the cdf technique to find the distribution of $U$. First, note that

$$1 < y < 5 \implies 5 < u < 53.$$

Therefore, the support of $U$ is $R_U = \{u : 5 < u < 53\}$. For $5 < u < 53$, the cdf of $U$ is

$$F_U(u) = P(U \leq u) = P(2Y^2 + 3 \leq u) = P \left( Y \leq \sqrt{\frac{u - 3}{2}} \right) = F_Y \left( \sqrt{\frac{u - 3}{2}} \right).$$

Recall the cdf of $Y \sim U(1, 5)$ is given by

$$F_Y(y) = \begin{cases} 
0, & y \leq 1 \\
\frac{y - 1}{4}, & 1 < y < 5 \\
1, & y \geq 5.
\end{cases}$$
Therefore, for $1 < y < 5 \iff 5 < u < 53$, the cdf of $U = h(Y) = 2Y^2 + 3$ is
\[ F_U(u) = F_Y \left( \sqrt{\frac{u - 3}{2}} \right) = \sqrt{\frac{u-3}{2}} - \frac{1}{4}. \]

For $5 < u < 53$, the pdf of $U$ is
\[ f_U(u) = \frac{d}{du} F_U(u) = \frac{1}{8} \left( \frac{u-3}{2} \right)^{-1/2} \left( \frac{1}{2} \right). \]

Summarizing,
\[ f_U(u) = \begin{cases} \frac{1}{16} \left( \frac{u-3}{2} \right)^{-1/2}, & 5 < u < 53 \\ 0, & \text{otherwise}. \end{cases} \]

The pdf of $U$ is shown at the top of the last page (right).

6.10. The support of $(Y_1, Y_2)$ is the set $R = \{(y_1, y_2) : 0 \leq y_2 \leq y_1 < \infty\}$; this is the triangular region shown below. The upper boundary line is $y_2 = y_1$. The joint pdf $f_{Y_1, Y_2}(y_1, y_2)$ is a three-dimensional function which takes the value $e^{-y_1}$ over this region and is otherwise equal to zero.

(a) We want to find the pdf of $U = Y_1 - Y_2$. We will use the cdf technique. First, observe that
\[ y_1 \geq y_2 \geq 0 \implies u = h(y_1, y_2) = y_1 - y_2 \geq 0. \]

Therefore, the support of $U = h(Y_1, Y_2) = Y_1 - Y_2$ is $R_U = \{u : u \geq 0\}$. For $u \geq 0$, the cdf of $U$ is
\[ F_U(u) = P(U \leq u) = P(Y_1 - Y_2 \leq u) = \int \int_{(y_1, y_2) \in B} f_{Y_1, Y_2}(y_1, y_2) dy_1 dy_2 = \int \int_{(y_1, y_2) \in B} e^{-y_1} dy_1 dy_2, \]

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where the set $B = \{(y_1, y_2) : y_1 \geq 0, y_2 \geq 0, y_1 - y_2 \leq u\}$. The region $B$ is shown at the top of this page. Note that the boundary of $B$ is

$$y_1 - y_2 = u \implies y_2 = y_1 - u,$$

a linear function of $y_1$ with slope 1 and intercept $-u$. The limits in the double integral (on the preceding page) come from this picture.

For $u \geq 0$, the cdf of $U$ is

$$F_U(u) = P(U \leq u) = \int_{y_2=0}^{\infty} \int_{y_1=y_2}^{\infty} e^{-y_1} dy_1 dy_2 = \int_{y_2=0}^{\infty} \left( e^{-y_1} \right)_{y_1=y_2} dy_2 = \int_{y_2=0}^{\infty} \left[ e^{-y_2} - e^{-(y_2+u)} \right] dy_2 = \int_{y_2=0}^{\infty} e^{-y_2} dy_2 - \int_{y_2=0}^{\infty} e^{-(y_2+u)} dy_2.$$

The first integral above is 1 because $e^{-y_2}$ is the exponential(1) pdf and we are integrating it over $(0, \infty)$. The second integral is

$$\int_{y_2=0}^{\infty} e^{-(y_2+u)} dy_2 = -e^{-(y_2+u)} \bigg|_{y_2=0}^{\infty} = 0 + e^{-u} = e^{-u}.$$

Therefore, for $u \geq 0$, the cdf of $U$ is $F_U(u) = 1 - e^{-u}$. Summarizing,

$$F_U(u) = \begin{cases} 0, & u < 0 \\ 1 - e^{-u}, & u \geq 0. \end{cases}$$
We recognize this as an exponential(1) cdf; i.e., \( U = Y_1 - Y_2 \sim \text{exponential}(1) \). For \( u \geq 0 \), the pdf of \( U \) is
\[
f_U(u) = \frac{d}{du} F_U(u) = \frac{d}{du} (1 - e^{-u}) = e^{-u}.
\]
Summarizing,
\[
f_U(u) = \begin{cases} e^{-u}, & u \geq 0 \\ 0, & \text{otherwise}. \end{cases}
\]
This is the pdf of \( U \sim \text{exponential}(1) \); i.e., an exponential pdf with mean \( \beta = 1 \).

(b) Based on our knowledge of the exponential distribution, we know
\[
E(U) = 1 \quad \text{and} \quad V(U) = 1.
\]
Comparing with Exercise 5.108, these are the same answers you would get if you calculated
\[
E(Y_1 - Y_2) \quad \text{and} \quad V(Y_1 - Y_2)
\]
by using the joint pdf of \( Y_1 \) and \( Y_2 \). To find \( E(Y_1 - Y_2) \), we would calculate
\[
E(Y_1 - Y_2) = \int \int (y_1 - y_2) f_{Y_1,Y_2}(y_1,y_2) dy_1 dy_2 = \int_{y_1=0}^{\infty} \int_{y_2=y_1}^{\infty} (y_1 - y_2)e^{-y_1} dy_2 dy_1.
\]
To get \( V(Y_1 - Y_2) \), we could first calculate
\[
E[(Y_1 - Y_2)^2] = \int \int (y_1 - y_2)^2 f_{Y_1,Y_2}(y_1,y_2) dy_1 dy_2 = \int_{y_1=0}^{\infty} \int_{y_2=y_1}^{\infty} (y_1 - y_2)^2 e^{-y_1} dy_2 dy_1
\]
and then use the variance computing formula
\[
V(Y_1 - Y_2) = E[(Y_1 - Y_2)^2] - [E(Y_1 - Y_2)]^2.
\]
We could also calculate \( V(Y_1 - Y_2) \) by using
\[
V(Y_1 - Y_2) = V(Y_1) + V(Y_2) - 2 \text{Cov}(Y_1,Y_2).
\]
As an exercise, try to calculate \( E(U) = E(Y_1 - Y_2) \) and \( V(U) = V(Y_1 - Y_2) \) by doing what is described above. It will be a lot of work, but it is a good review of Chapter 5 calculations. From the Law of the Unconscious Statistician, we know \( E(Y_1 - Y_2) = 1 \) and \( V(Y_1 - Y_2) = 1 \).

6.14. Because \( Y_1 \) and \( Y_2 \) are independent (by assumption), the joint pdf of \( Y_1 \) and \( Y_2 \), for \( 0 \leq y_1 \leq 1 \) and \( 0 \leq y_2 \leq 1 \), is given by
\[
f_{Y_1,Y_2}(y_1,y_2) = f_{Y_1}(y_1) f_{Y_2}(y_2) = 6y_1(1 - y_1) \times 3y_2^2 = 18y_1(1 - y_1)y_2^2.
\]
Summarizing,
\[
f_{Y_1,Y_2}(y_1,y_2) = \begin{cases} 18y_1(1 - y_1)y_2^2, & 0 \leq y_1 \leq 1, \ 0 \leq y_2 \leq 1 \\ 0, & \text{otherwise}. \end{cases}
\]
The support of \((Y_1,Y_2)\) is the set \( R = \{(y_1,y_2) : 0 \leq y_1 \leq 1, \ 0 \leq y_2 \leq 1\} \); i.e., the unit square. This region is shown at the top of the next page (left). The joint pdf \( f_{Y_1,Y_2}(y_1,y_2) \) is a three-dimensional function which takes the value \( 18y_1(1 - y_1)y_2^2 \) over this region and is otherwise equal to zero.
We want to find the pdf of $U = Y_1Y_2$. We will use the cdf technique. First, observe that

$$0 \leq y_1 \leq 1, \ 0 \leq y_2 \leq 1 \implies u = h(y_1, y_2) = y_1y_2 \in [0, 1].$$

Therefore, the support of $U = h(Y_1, Y_2) = Y_1Y_2$ is $R_U = \{u : 0 \leq u \leq 1\}$. For $0 \leq u \leq 1$, the cdf of $U$ is

$$F_U(u) = P(U \leq u) = P(Y_1Y_2 \leq u) = \int \int_{(y_1, y_2) \in B} f_{Y_1, Y_2}(y_1, y_2)dy_1dy_2 = \int \int_{(y_1, y_2) \in B} 18y_1(1 - y_1)y_2^2dy_1dy_2,$$

where the set $B = \{(y_1, y_2) : 0 \leq y_1 \leq 1, \ 0 \leq y_2 \leq 1, \ y_1y_2 \leq u\}$. The region $B$ is shown at the top of this page (right). Note that the boundary of $B$ is

$$y_1y_2 = u \implies y_2 = \frac{u}{y_1}$$

a decreasing curvilinear function of $y_1$. The limits in the double integral above come from this picture.

Note: In this situation, it is much easier to integrate the joint pdf $f_{Y_1, Y_2}(y_1, y_2)$ over the complement of the shaded region above (right). That is,

$$F_U(u) = P(U \leq u) = P(Y_1Y_2 \leq u) = 1 - P(Y_1Y_2 > u).$$

The reason it is easier is that we can get the limits of integration easier (i.e., by integrating over the white region instead of the grey region).
For $0 \leq u \leq 1$, we have

\[
P(Y_1 Y_2 > u) = \int_{y_2 = u}^{1} \int_{y_1 = u/y_2}^{1} 18y_1(1 - y_1)y_2^2 dy_1 dy_2
\]
\[
= \int_{y_2 = u}^{1} 18y_2^3 \left( \frac{y_1^3}{2} - \frac{y_1^3}{3} \right) \bigg|_{y_1 = u/y_2}^{1} dy_2
\]
\[
= \int_{y_2 = u}^{1} 3y_2^2 \left( 3y_1^2 - 2y_1^3 \right) \bigg|_{y_1 = u/y_2}^{1} dy_2
\]
\[
= \int_{y_2 = u}^{1} 3y_2^2 \left( 1 - \frac{3u^2}{y_2^2} + \frac{2u^3}{y_2^3} \right) dy_2
\]
\[
= \int_{y_2 = u}^{1} \left( 3y_2^2 - 9u^2 + 6u^3 \ln y_2 \right) dy_2
\]
\[
= \left( y_2^3 - 9u^2 y_2 + 6u^3 \ln y_2 \right) \bigg|_{y_2 = u}^{1}
\]
\[
= 1 - 9u^2 + 0 - u^3 + 9u^3 - 6u^3 \ln u = 1 - 9u^2 + 8u^3 - 6u^3 \ln u.
\]

Therefore, for $0 \leq u \leq 1$, we have

\[
F_U(u) = 1 - P(Y_1 Y_2 > u) = 1 - (1 - 9u^2 + 8u^3 - 6u^3 \ln u) = 9u^2 - 8u^3 + 6u^3 \ln u.
\]

Summarizing,

\[
F_U(u) = \begin{cases} 
0, & u < 0 \\
9u^2 - 8u^3 + 6u^3 \ln u, & 0 \leq u \leq 1 \\
1, & u > 1.
\end{cases}
\]

For $0 \leq u \leq 1$, the pdf of $U$ is

\[
f_U(u) = \frac{d}{du} F_U(u) = \frac{d}{du} (9u^2 - 8u^3 + 6u^3 \ln u)
\]
\[
= 18u - 24u^2 + (18u^2 \ln u + 6u^2)
\]
\[
= 18u - 18u^2 + 18u^2 \ln u = 18u(1 - u + u \ln u).
\]

Summarizing,

\[
f_U(u) = \begin{cases} 
18u(1 - u + u \ln u), & 0 \leq u \leq 1 \\
0, & \text{otherwise}.
\end{cases}
\]

A graph of this pdf appears at the top of the next page. I used R to ensure this pdf is valid; i.e., it integrates to 1.

\[
> \text{integrand} <- \text{function}(u)\{18*u*(1-u+u*log(u))\}
\]
\[
> \text{integrate(integrand,lower=0,upper=1)}
\]
\[
1 \text{ with absolute error < 3.7e-05}
\]

6.19. This exercise asks you establish a relationship between two new families of distributions; the power family and the Pareto family of distributions. Suppose $Y \sim \text{Pareto}(\alpha, \beta)$, where $\alpha > 0$ and $\beta > 0$. The cdf of $Y$ is given in Exercise 6.18; it is

\[
F_Y(y) = \begin{cases} 
0, & y < \beta \\
1 - \left( \frac{\beta}{y} \right)^\alpha, & y \geq \beta.
\end{cases}
\]
Consider the function

\[ X = h(Y) = \frac{1}{Y} \]

Note that

\[ y \geq \beta > 0 \implies 0 \leq \frac{1}{y} \leq \frac{1}{\beta}. \]

Therefore, the support of \( X \) is \( R_X = \{ x : 0 \leq x \leq 1/\beta \} \). For \( 0 \leq x \leq 1/\beta \), the cdf of \( X \) is

\[
F_X(x) = P(X \leq x) = P \left( \frac{1}{Y} \leq x \right) = P \left( Y \geq \frac{1}{x} \right) \\
= 1 - P \left( Y \leq \frac{1}{x} \right) \\
= 1 - F_Y \left( \frac{1}{x} \right) = 1 - \left[ 1 - \left( \frac{\beta}{1/x} \right)^\alpha \right] = (x\beta)^\alpha.
\]

Summarizing,

\[
F_X(x) = \begin{cases} 
0, & x < 0 \\
(x\beta)^\alpha, & 0 \leq x \leq 1/\beta \\
1, & x > 1/\beta.
\end{cases}
\]

Letting \( \theta = 1/\beta \), we have

\[
F_X(x) = \begin{cases} 
0, & x < 0 \\
\left( \frac{x}{\theta} \right)^\alpha, & 0 \leq x \leq \theta \\
1, & x > \theta.
\end{cases}
\]

That is, \( X \) follows a power family distribution with parameters \( \alpha \) and \( \theta = 1/\beta \).