6.34. A Rayleigh random variable \( Y \) has pdf
\[
f_Y(y) = \begin{cases} 
\frac{2y}{\theta} e^{-y^2/\theta}, & y > 0 \\
0, & \text{otherwise}
\end{cases}
\]
Note that this pdf arises when
\[
f_Y(y) = \begin{cases} 
\frac{m}{\theta} y^{m-1} e^{-y^m/\theta}, & y > 0 \\
0, & \text{otherwise}
\end{cases}
\]
and \( m = 2 \). In other words, the Rayleigh(\( \theta \)) distribution is a special case of the Weibull(\( m, \theta \)) distribution with \( m = 2 \). We proved the general result
\[
Y \sim \text{Weibull}(m, \theta) \implies U = h(Y) = Y^m \sim \text{exponential}(\theta)
\]
in Exercise 6.26 (HW1) by using the transformation method. Therefore, arguing
\[
Y \sim \text{Rayleigh}(\theta) \implies U = h(Y) = Y^2 \sim \text{exponential}(\theta)
\]
is a special case of this general argument when \( m = 2 \). For fun, let’s prove this result (when \( m = 2 \)) by using the cdf technique and the mgf technique (in other words, all three methods “work” in this instance).

**CDF technique:** Let’s first derive the cdf of \( Y \sim \text{Rayleigh}(\theta) \). When \( y \leq 0 \), the cdf
\[
F_Y(y) = \int_{-\infty}^{y} f_Y(t) dt = \int_{-\infty}^{y} 0 dt = 0.
\]
For \( y > 0 \), the cdf
\[
F_Y(y) = \int_{-\infty}^{y} f_Y(t) dt = \int_{-\infty}^{0} 0 dt + \int_{0}^{y} \frac{2t}{\theta} e^{-t^2/\theta} dt = \int_{0}^{y} \frac{2t}{\theta} e^{-t^2/\theta} dt.
\]
In the last integral, let
\[
u = t^2 \implies du = 2t dt.
\]
The limits on the integral change under this transformation. Note that
\[
t : 0 \to y \implies u : 0 \to y^2.
\]
Therefore, for \( y > 0 \),
\[
F_Y(y) = \int_{0}^{y} \frac{2t}{\theta} e^{-t^2/\theta} dt = \int_{0}^{y^2} \frac{1}{\theta} e^{-u/\theta} du
\]
\[
= \int_{0}^{y^2} \frac{1}{\theta} e^{-u/\theta} du
\]
\[
= \frac{1}{\theta} \left( -\theta e^{-u/\theta} \right)_{0}^{y^2} = e^{-u/\theta} \bigg|_{0}^{y^2} = 1 - e^{-y^2/\theta}.
\]
Summarizing,
\[
F_Y(y) = \begin{cases} 
0, & y \leq 0 \\
1 - e^{-y^2/\theta}, & y > 0
\end{cases}
\]
We are now ready to derive the cdf of \( U = Y^2 \). For \( u > 0 \), it is
\[
F_U(u) = P(U \leq u) = P(Y^2 \leq u) = P(Y \leq \sqrt{u}) = F_Y(\sqrt{u}) = 1 - e^{-（\sqrt{u}/\theta)^2} = 1 - e^{-u/\theta}.
\]
Summarizing,
\[
F_U(u) = \begin{cases} 
0, & u \leq 0 \\
1 - e^{-u/\theta}, & u > 0.
\end{cases}
\]
We recognize this as the cdf of \( U \sim \text{exponential}(\theta) \). Therefore, we are done.

**MGF technique:** We derive the mgf of \( U = Y^2 \) and show that it matches the mgf of an exponential random variable with mean \( \theta \). The mgf of \( U \) is
\[
m_U(t) = E(e^{tU}) = E(e^{tY^2}) = \int_0^{\infty} e^{ty^2} \times \frac{2}{\theta} e^{-y^2/\theta} dy = \int_0^{\infty} \frac{2y}{\theta} e^{ty^2-y^2/\theta} dy.
\]
In the exponent of \( e^{ty^2-y^2/\theta} \), write
\[
ty^2 - \frac{y^2}{\theta} = -y^2 \left( \frac{1}{\theta} - t \right) = -y^2 \left( \frac{1}{\theta} - t \right)^{-1} = -y^2/\eta,
\]
where \( \eta = (\frac{1}{\theta} - t)^{-1} \). Therefore, the last integral becomes
\[
m_U(t) = \int_0^{\infty} \frac{2y}{\theta} e^{-y^2/\eta} dy = \int_0^{\infty} \frac{2y}{\theta} e^{-y^2/\eta} dy.
\]
Now, let
\[
u = y^2 \implies du = 2y dy.
\]
The limits on the integral do not change under this transformation. Note that
\[
y : 0 \to \infty \implies u : 0 \to \infty.
\]
Therefore,
\[
m_U(t) = \int_0^{\infty} \frac{2y}{\theta} e^{-u/\eta} du = \frac{1}{\theta} \eta^{-1}(e^{-u/\eta}) \bigg|_0^\infty = \eta \left( \frac{1}{\theta} - t \right) = \frac{\eta}{\theta},
\]
provided that
\[
\eta > 0 \iff t < \frac{1}{\theta}.
\]
Therefore, for \( t < 1/\theta \), we have
\[
m_U(t) = \frac{1}{\theta} \left( \frac{1}{\theta} - t \right) = \frac{\theta}{1 - \theta t} = \frac{1}{1 - \theta t}.
\]
We recognize this mgf as the mgf of an exponential random variable with mean \( \theta \). Because mgfs are unique, we know \( U \sim \text{exponential}(\theta) \).
(b) In HW1, we derived the mean and variance of $Y \sim \text{Weibull}(m, \theta)$ to be

$$E(Y) = \theta \frac{1}{m} \Gamma \left( \frac{1}{m} + 1 \right)$$

$$V(Y) = \theta \frac{2}{m} \left\{ \Gamma \left( \frac{2}{m} + 1 \right) - \left[ \Gamma \left( \frac{1}{m} + 1 \right) \right]^2 \right\}.$$ 

Therefore, for $Y \sim \text{Raleigh}(\theta)$, put in $m = 2$ and we get

$$E(Y) = \theta \frac{1}{2} \Gamma \left( \frac{3}{2} \right) = \theta \frac{1}{2} \Gamma \left( \frac{1}{2} \right) = \frac{\sqrt{\pi \theta}}{2}.$$ 

and

$$V(Y) = \theta \left\{ \Gamma \left( \frac{2}{2} + 1 \right) - \left[ \Gamma \left( \frac{1}{2} + 1 \right) \right]^2 \right\} = \theta \left( \Gamma(2) - \left[ \Gamma \left( \frac{3}{2} \right) \right]^2 \right) = \theta \left( 1 - \frac{\pi}{4} \right).$$

6.40. We know that $Y \sim \mathcal{N}(0, 1) \implies Y^2 \sim \chi^2(1)$. Therefore, $Y_1^2$ and $Y_2^2$ are independent random variables, both distributed as $\chi^2(1)$. Recall the $\chi^2(1)$ mgf is given by

$$m_{Y^2}(t) = \left( \frac{1}{1 - 2t} \right)^{1/2},$$

for $t < 1/2$. Therefore, the mgf of $U = Y_1^2 + Y_2^2$ is

$$m_U(t) = m_{Y_1^2}(t)m_{Y_2^2}(t) = \left( \frac{1}{1 - 2t} \right)^{1/2} \left( \frac{1}{1 - 2t} \right)^{1/2} = \left( \frac{1}{1 - 2t} \right)^{2/2}.$$ 

We recognize this mgf as the mgf of a $\chi^2$ random variable with 2 degrees of freedom. Because mgfs are unique, we know $U = Y_1^2 + Y_2^2 \sim \chi^2(2)$; i.e., the degrees of freedom simply “add.”

6.42. The weight capacity $Y_1 \sim \mathcal{N}(5000, 300^2)$. The load $Y_2 \sim \mathcal{N}(4000, 400^2)$. The elevator will be overloaded when $Y_1 < Y_2$; i.e., when $U = Y_1 - Y_2 < 0$. Therefore, we want to find $P(Y_1 < Y_2) = P(U < 0)$.

In Example 6.13 (notes), we proved that linear combinations of mutually independent normal random variables are normally distributed; i.e.,

$$U = \sum_{i=1}^{n} a_i Y_i \sim \mathcal{N} \left( \sum_{i=1}^{n} a_i \mu_i, \sum_{i=1}^{n} a_i^2 \sigma_i^2 \right).$$

Note that

$$U = Y_1 - Y_2$$

is a special case of the linear combination above with $n = 2$, $a_1 = 1$, and $a_2 = -1$. Therefore, we know $U = Y_1 - Y_2$ is normally distributed with mean

$$a_1 \mu_1 + a_2 \mu_2 = 1(5000) + (-1)(4000) = 1000$$
and variance
\[ a_1^2\sigma_1^2 + a_2^2\sigma_2^2 = 1^2(300^2) + (-1)^2(400^2) = 500^2. \]
That is, \( U \sim \mathcal{N}(1000, 500^2) \). We can calculate \( P(U < 0) \) in R; note that
\[
> \text{pnorm}(0, 1000, 500)
[1] 0.02275013
\]
Therefore,
\[
P(Y_1 < Y_2) = P(U < 0) \approx 0.0228.
\]
The pdf of \( U \sim \mathcal{N}(1000, 500^2) \) is shown at the top of this page with the probability \( P(U < 0) \) shown shaded.

6.45. We are given
\[
Y_1 = \text{amount of sand (in yards)} \sim \mathcal{N}(10, 0.5^2) \\
Y_2 = \text{amount of cement (in 100s lbs)} \sim \mathcal{N}(4, 0.2^2).
\]
The total cost is
\[
U = 100 + 7Y_1 + 3Y_2.
\]
We are told to assume that \( Y_1 \) and \( Y_2 \) are independent. Under this assumption,
\[
7Y_1 + 3Y_2
\]
is a linear combination of independent normally distributed random variables with \( n = 2, a_1 = 7, \) and \( a_2 = 3 \). Therefore, it too is normally distributed with mean
\[
a_1\mu_1 + a_2\mu_2 = 7(10) + 3(4) = 82
\]
and variance
\[ a_1^2 \sigma_1^2 + a_2^2 \sigma_2^2 = 7^2(0.5^2) + 3^2(0.2^2) = 12.61. \]
That is,
\[ 7Y_1 + 3Y_2 \sim \mathcal{N}(82, 12.61). \]
Now, the additive constant 100 merely shifts the \( \mathcal{N}(82, 12.61) \) distribution 100 units to the right; therefore,
\[ U = 100 + 7Y_1 + 3Y_2 \sim \mathcal{N}(182, 12.61). \]
\textbf{Note:} If you dislike the previous argument, you can derive the mgf of \( U = 100 + 7Y_1 + 3Y_2 \) directly and show that it matches the mgf of a \( \mathcal{N}(182, 12.61) \) random variable. We do this now:

\[
m_U(t) = E(e^{tU}) = E[e^{(100+7Y_1+3Y_2)t}] = E(e^{100t}e^{7tY_1}e^{3tY_2}) = e^{100t}m_{Y_1}(7t)m_{Y_2}(3t),
\]
where \( m_{Y_1}(t) \) is the \( \mathcal{N}(10, 0.5^2) \) mgf and where \( m_{Y_2}(t) \) is the \( \mathcal{N}(4, 0.2^2) \) mgf. We have

\[
m_{Y_1}(t) = \exp\left[10t + \frac{(0.5^2)t^2}{2}\right] \implies m_{Y_1}(7t) = \exp\left[70t + \frac{49(0.5^2)t^2}{2}\right]
\]
and

\[
m_{Y_2}(t) = \exp\left[4t + \frac{(0.2^2)t^2}{2}\right] \implies m_{Y_1}(3t) = \exp\left[12t + \frac{9(0.2^2)t^2}{2}\right]
\]
Therefore,

\[
m_U(t) = e^{100t}m_{Y_1}(7t)m_{Y_2}(3t) = \exp(100t)\exp\left[70t + \frac{49(0.5^2)t^2}{2}\right]\exp\left[12t + \frac{9(0.2^2)t^2}{2}\right]
\]
\[
= \exp\left\{182t + \frac{49(0.5^2) + 9(0.2^2)}{2}\right\}
\]
\[
= \exp\left(182t + \frac{12.61t^2}{2}\right).
\]
We recognize this as the mgf of a normal random variable with mean \( \mu = 182 \) and variance \( \sigma^2 = 12.61 \). Because mgfs are unique, we know that \( U \sim \mathcal{N}(182, 12.61) \). Now, the bidding problem being asked is this. What should the manager bid on the job so that the total cost \( U \) will exceed the bid with probability 0.01? Let \( b \) denote the bid the manager makes. S/he wants to select \( b \) so that

\[ P(U > b) = 0.01. \]
In other words, s/he wants to bid the 99th percentile (\( p = 0.99 \) quantile) of \( U \sim \mathcal{N}(182, 12.61) \). In R, we have

\[
> \text{qnorm}(0.99, 182, \text{sqrt}(12.61))
\]
\[
[1] 190.261
\]
Therefore, if s/he sets the bid at \( b = 190.261 \), then the total cost \( U \) will exceed this value with probability 0.01. See the figure at the top of the next page.
Remark: We are asked to comment on whether the amount of sand required and the amount of cement required for the construction job are independent; i.e., if it is reasonable to assume $Y_1$ and $Y_2$ are independent. On practical grounds, they probably aren’t; in fact, we would expect them to be positively correlated (i.e., the more sand required for the construction job, the more cement will be required). Therefore, the solution we obtained ($b = 190.261$) isn’t 100 percent correct if $Y_1$ and $Y_2$ are in fact correlated. However, we made the independence assumption so that we could get a solution. This is commonly done in statistical problems—we sometimes are forced to make simplifying assumptions so that we can get an answer. If we wanted to solve $P(U > b) = 0.01$ while allowing for dependence between $Y_1$ and $Y_2$, we would have to know the covariance of $Y_1$ and $Y_2$. If we knew this, then we could recalculate the distribution of $U$. It is still normal with mean $E(U) = 182$, but the variance would change as follows:

$$V(U) = V(100 + 7Y_1 + 3Y_2) = V(7Y_1 + 3Y_2) = 49V(Y_1) + 9V(Y_2) + 2(7)(3) \text{Cov}(Y_1, Y_2).$$

6.48. In this problem, we are given $Y_1 \sim \mathcal{N}(0, 1)$, $Y_2 \sim \mathcal{N}(0, 1)$, and $Y_1$ and $Y_2$ are independent. We want to find the distribution of

$$U = \sqrt{Y_1^2 + Y_2^2}.$$

From Exercise 6.40, we already know

$$V = Y_1^2 + Y_2^2 \sim \chi^2(2).$$

Therefore, all we have to do is find the pdf of $U = h(V) = \sqrt{V}$, where $V \sim \chi^2(2) \overset{d}{=} \text{gamma}(1, 2)$.

The pdf of $V$, for $v > 0$, is

$$f_V(v) = \frac{1}{\Gamma(1/2)^2}v^{1/2-1}e^{-v/2} = \frac{1}{2}e^{-v/2},$$

which is the exponential(2) pdf with mean $\beta = 2$. In other words, the $\chi^2(2)$ pdf, the gamma(1, 2) pdf, and the exponential(2) pdf are all the same pdf! Interesting!!
To find the pdf of \( U = h(V) = \sqrt{V} \), we will use the transformation method. Note that
\[
v > 0 \implies u = \sqrt{v} > 0.
\]
Therefore, the support of \( U \) is \( R_U = \{u : u > 0\} \). Also, the function \( u = h(v) = \sqrt{v} \) is 1:1 over \( R_V = \{v : v > 0\} \), the support of \( V \). Therefore, we can use the transformation method.

The inverse transformation is found as follows:
\[
u = h(v) = \sqrt{v} \implies v = u^2 = h^{-1}(u).
\]
Also, the derivative of the inverse transformation is
\[
\frac{d}{du}h^{-1}(u) = \frac{d}{du}u^2 = 2u.
\]
Therefore, for \( u > 0 \), the pdf of \( U \) is
\[
f_U(u) = f_V(h^{-1}(u)) \left| \frac{d}{du}h^{-1}(u) \right| = \frac{1}{2}e^{-u^2/2} \times |2u| = ue^{-u^2/2}.
\]

Summarizing, the pdf of \( U = h(V) = \sqrt{V} \) is
\[
f_U(u) = \begin{cases} 
ue^{-u^2/2}, & u > 0 \\
0, & \text{otherwise}.
\end{cases}
\]

Comparing this pdf to the general form of the Weibull(\( m, \theta \)) pdf
\[
f_U(u) = \begin{cases} 
\frac{m}{\theta}u^{m-1}e^{-u^m/\theta}, & u > 0 \\
0, & \text{otherwise},
\end{cases}
\]
we see that \( U \sim \text{Weibull}(m = 2, \theta = 2) \). This pdf is shown above.
6.52. (a) We did this part in Example 6.11 of the notes. Suppose \( Y_1 \sim \text{Poisson}(\lambda_1) \) and \( Y_2 \sim \text{Poisson}(\lambda_2) \). If \( Y_1 \) and \( Y_2 \) are independent, the mgf of \( U = Y_1 + Y_2 \) is

\[
m_U(t) = m_{Y_1}(t)m_{Y_2}(t) = e^{\lambda_1(e^t-1)}e^{\lambda_2(e^t-1)} = e^{(\lambda_1+\lambda_2)(e^t-1)}.
\]

We recognize this as the mgf of a Poisson random variable with mean \( \lambda_1 + \lambda_2 \). Because mgfs are unique, we know that \( U \sim \text{Poisson}(\lambda_1 + \lambda_2) \). The pmf of \( U \) is

\[
p_U(u) = \begin{cases} 
\frac{(\lambda_1 + \lambda_2)^u e^{-(\lambda_1+\lambda_2)}}{u!}, & u = 0, 1, 2, \\ 
0, & \text{otherwise}.
\end{cases}
\]

(b) In this part, we want to find \( p_{Y_1|U}(y_1|m) \), the conditional pmf of \( Y_1 \), given \( U = Y_1 + Y_2 = m \). First note that if the sum \( U = Y_1 + Y_2 = m \), then the possible values of \( Y_1 \) are \( \{y_1 : y_1 = 0, 1, 2, ..., m\} \). Therefore, the conditional pmf \( p_{Y_1|U}(y_1|m) \) is nonzero for these values of \( y_1 \), and is otherwise equal to zero. Recall from STAT 511, the definition of a conditional pmf

\[
p_{Y_1|U}(y_1|m) = \frac{p_{Y_1,U}(y_1,m)}{p_U(m)} = \frac{P(Y_1 = y_1, U = m)}{P(U = m)}.
\]

We know

\[
P(U = m) = p_U(m) = \frac{(\lambda_1 + \lambda_2)^m e^{-(\lambda_1+\lambda_2)}}{m!}
\]

from part (a). How do we find the joint probability \( P(Y_1 = y_1, U = m) \)? We don’t have the joint pmf of \( Y_1 \) and \( U \), so it is not clear how to calculate this. The key is to note that

\[
\{Y_1 = y_1, U = m\} = \{Y_1 = y_1, Y_1 + Y_2 = m\} = \{Y_1 = y_1, Y_2 = m - y_1\}.
\]

Therefore,

\[
P(Y_1 = y_1, U = m) = P(Y_1 = y_1, Y_2 = m - y_1) \overset{Y_1 \perp \perp Y_2}{=} P(Y_1 = y_1)P(Y_2 = m - y_1).
\]

We can calculate these two probabilities because \( Y_1 \sim \text{Poisson}(\lambda_1) \) and \( Y_2 \sim \text{Poisson}(\lambda_2) \); that is,

\[
P(Y_1 = y_1) = \frac{\lambda_1^{y_1} e^{-\lambda_1}}{y_1!} \quad \text{and} \quad P(Y_2 = m - y_1) = \frac{\lambda_2^{m-y_1} e^{-\lambda_2}}{(m - y_1)!}.
\]

Therefore,

\[
p_{Y_1|U}(y_1|m) = \frac{P(Y_1 = y_1)P(Y_2 = m - y_1)}{P(U = m)} \frac{\lambda_1^{y_1} e^{-\lambda_1}}{y_1!} \frac{\lambda_2^{m-y_1} e^{-\lambda_2}}{(m - y_1)!}
\]

\[
= \frac{m!}{y_1!(m - y_1)!} \frac{\lambda_1^{y_1}}{(\lambda_1 + \lambda_2)^y_1} \frac{\lambda_2^{m-y_1}}{(\lambda_1 + \lambda_2)^{m-y_1}}
\]

\[
= \left( \frac{m}{y_1} \right) \left( \frac{\lambda_1}{\lambda_1 + \lambda_2} \right)^{y_1} \left( \frac{\lambda_2}{\lambda_1 + \lambda_2} \right)^{m-y_1} = \left( \frac{\lambda_1}{\lambda_1 + \lambda_2} \right)^{y_1} \left( \frac{\lambda_2}{\lambda_1 + \lambda_2} \right)^{m-y_1}.
\]
Summarizing,

\[
p_{Y_1|U}(y_1|m) = \begin{cases} 
  \left( \frac{\lambda_1}{\lambda_1 + \lambda_2} \right)^{y_1} \left( 1 - \frac{\lambda_1}{\lambda_1 + \lambda_2} \right)^{m-y_1}, & y_1 = 0, 1, 2, \ldots, m \\
  0, & \text{otherwise.}
\end{cases}
\]

We recognize this as the pmf of a binomial random variable with number of trials \(m\) and success probability \(p = \frac{\lambda_1}{\lambda_1 + \lambda_2}\).

Therefore, we have shown

\[Y_1 \sim \text{Poisson}(\lambda_1), Y_2 \sim \text{Poisson}(\lambda_2), Y_1 \perp Y_2 \implies Y_1|Y_1 + Y_2 = m \sim \text{b}(m, \frac{\lambda_1}{\lambda_1 + \lambda_2}).\]

6.57. We are given

\[Y_1 \sim \text{gamma}(\alpha_1, \beta), Y_2 \sim \text{gamma}(\alpha_2, \beta), \ldots, Y_n \sim \text{gamma}(\alpha_n, \beta)\]

and \(Y_1, Y_2, \ldots, Y_n\) are mutually independent. We want to find the distribution of

\[U = Y_1 + Y_2 + \cdots + Y_n.\]

Whenever you are asked to find the distribution of the sum of mutually independent random variables, try the mgf method. The mgf of the sum \(U\) is

\[
m_U(t) = m_{Y_1}(t)m_{Y_2}(t)\cdots m_{Y_n}(t) = \left( \frac{1}{1 - \beta t} \right)^{\alpha_1} \times \left( \frac{1}{1 - \beta t} \right)^{\alpha_2} \times \cdots \times \left( \frac{1}{1 - \beta t} \right)^{\alpha_n} = \left( \frac{1}{1 - \beta t} \right)^{\sum \alpha_i}.\]

We recognize this as the mgf of a gamma random variable with shape parameter \(\sum \alpha_i\) and scale parameter \(\beta\). Because mgfs are unique, we know

\[U = Y_1 + Y_2 + \cdots + Y_n \sim \text{gamma}(\sum \alpha_i, \beta).\]

6.59. We are given \(Y_1 \sim \chi^2(\nu_1), Y_2 \sim \chi^2(\nu_2)\), and \(Y_1\) and \(Y_2\) are independent. We want to find the distribution of \(U = Y_1 + Y_2\). Use the mgf method. The mgf of the sum \(U\) is

\[
m_U(t) = m_{Y_1}(t)m_{Y_2}(t) = \left( \frac{1}{1 - 2t} \right)^{\nu_1/2} \left( \frac{1}{1 - 2t} \right)^{\nu_2/2} = \left( \frac{1}{1 - 2t} \right)^{\nu_1/2 + \nu_2/2}.
\]

We recognize this as the mgf of a \(\chi^2\) random variable with degrees of freedom \(\nu_1 + \nu_2\). Because mgfs are unique, we know \(U = Y_1 + Y_2 \sim \chi^2(\nu_1 + \nu_2)\).

Note: See how easy the mgf method is? As an exercise, try to redo Exercise 6.59 by using the cdf method; i.e., derive \(F_U(u) = P(U \leq u)\) directly and then take derivatives. You should get the \(\chi^2(\nu_1 + \nu_2)\) pdf. This argument is much harder, but it still should work.
6.63. The authors have already done the bivariate transformation for us. Starting with \( Y_1 \sim \text{exponential}(\beta) \), \( Y_2 \sim \text{exponential}(\beta) \), and \( Y_1 \perp Y_2 \), the authors show the joint distribution of

\[
U_1 = \frac{Y_1}{Y_1 + Y_2} \quad \text{and} \quad U_2 = Y_1 + Y_2
\]

is

\[
f_{U_1, U_2}(u_1, u_2) = \begin{cases} \frac{1}{\beta^2} u_2 e^{-u_2/\beta}, & 0 < u_1 < 1, \ u_2 > 0 \\ 0, & \text{otherwise.} \end{cases}
\]

Go through the bivariate transformation again and re-derive this yourself for practice. Note the support of \((U_1, U_2)\)

\[
R_{U_1, U_2} = \{(u_1, u_2) : 0 < u_1 < 1, \ u_2 > 0\}.
\]

This region is shown above. The joint pdf \( f_{U_1, U_2}(u_1, u_2) \) is a three-dimensional function which takes the value \( \frac{1}{\beta^2} u_2 e^{-u_2/\beta} \) over this region and is otherwise equal to zero.

(a) To find the marginal distribution of \( U_1 \), we integrate the joint pdf \( f_{U_1, U_2}(u_1, u_2) \) over \( u_2 \).

For \( 0 < u_1 < 1 \), we have

\[
f_{U_1}(u_1) = \int_{u_2=0}^{\infty} f_{U_1, U_2}(u_1, u_2) du_2 = \int_{u_2=0}^{\infty} \frac{1}{\beta^2} u_2 e^{-u_2/\beta} du_2 = 1,
\]

because \( \frac{1}{\beta^2} u_2 e^{-u_2/\beta} \) is the gamma(2, \( \beta \)) pdf and we are integrating over \((0, \infty)\). We have shown

\[
f_{U_1}(u_1) = \begin{cases} 1, & 0 < u_1 < 1 \\ 0, & \text{otherwise.} \end{cases}
\]

We recognize this as the \( U(0, 1) \) pdf; i.e., \( U_1 \sim U(0, 1) \).
(b) To find the marginal distribution of $U_2$, we integrate the joint pdf $f_{U_1,U_2}(u_1,u_2)$ over $u_1$. For $u_2 > 0$, we have

$$f_{U_2}(u_2) = \int_{u_1=0}^{1} f_{U_1,U_2}(u_1,u_2) \, du_1 = \int_{u_1=0}^{1} \frac{1}{\beta^2} u_2 e^{-u_2/\beta} \, du_1$$

$$= \frac{1}{\beta^2} u_2 e^{-u_2/\beta} \int_{u_1=0}^{1} 1 \, du_1 = \frac{1}{\beta^2} u_2 e^{-u_2/\beta}.$$

We have shown

$$f_{U_2}(u_2) = \begin{cases} 
\frac{1}{\beta^2} u_2 e^{-u_2/\beta}, & u_2 > 0 \\
0, & \text{otherwise.}
\end{cases}$$

We recognize this as the gamma$(2, \beta)$ pdf; i.e., $U_2 \sim \text{gamma}(2, \beta)$.

(c) Note that we can write

$$f_{U_1,U_2}(u_1,u_2) = \frac{1}{\beta^2} u_2 e^{-u_2/\beta} = 1 \times \frac{1}{\beta^2} u_2 e^{-u_2/\beta} = f_{U_1}(u_1) f_{U_2}(u_2).$$

Because the joint pdf can be written as the product of the marginal pdfs, we know $U_1 \perp \perp U_2$.

6.68. We start with the random variables $Y_1$ and $Y_2$, whose joint pdf is

$$f_{Y_1,Y_2}(y_1,y_2) = \begin{cases} 
8y_1 y_2, & 0 \leq y_1 \leq y_2 \leq 1 \\
0, & \text{otherwise.}
\end{cases}$$

Note the support of $(Y_1, Y_2)$ is

$$R_{Y_1,Y_2} = \{(y_1, y_2) : 0 \leq y_1 \leq y_2 \leq 1\}.$$ 

The graph of $R_{Y_1,Y_2}$ is shown at the top of the next page (left). The joint pdf $f_{Y_1,Y_2}(y_1,y_2)$ is a three-dimensional function which takes the value $8y_1y_2$ over this triangular region and is otherwise equal to zero.

Our goal is to find the joint pdf of

$$U_1 = h_1(Y_1, Y_2) = \frac{Y_1}{Y_2}$$

$$U_2 = h_2(Y_1, Y_2) = Y_2.$$

We use a bivariate transformation. We first find the support of $(U_1, U_2)$. Note that

$$0 \leq y_1 \leq y_2 \leq 1 \implies u_1 = \frac{y_1}{y_2} \in [0, 1]$$

and $0 \leq u_2 = y_2 \leq 1$. Therefore, the support of $(U_1, U_2)$ is

$$R_{U_1,U_2} = \{(u_1, u_2) : 0 \leq u_1 \leq 1, \ 0 \leq u_2 \leq 1\}.$$ 

The graph of $R_{U_1,U_2}$ is shown at the top of the next page (right).
To verify the transformation above is one-to-one, we show $h(y_1, y_2) = h(y_1^*, y_2^*) \implies y_1 = y_1^*$ and $y_2 = y_2^*$, where

$$h \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} h_1(y_1, y_2) \\ h_2(y_1, y_2) \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Suppose $h(y_1, y_2) = h(y_1^*, y_2^*)$. Clearly $y_2 = y_2^*$. Then the first equation implies $y_1 = y_1^*$. Therefore the transformation is one to one.

The inverse transformation is found by solving

$$u_1 = \frac{y_1}{y_2} \quad u_2 = \frac{y_2}{y_2}$$

for $y_1 = h_1^{-1}(u_1, u_2)$ and $y_2 = h_2^{-1}(u_1, u_2)$. Straightforward algebra shows

$$y_1 = h_1^{-1}(u_1, u_2) = u_1 u_2 \quad y_2 = h_2^{-1}(u_1, u_2) = u_2.$$

The Jacobian is

$$J = \begin{vmatrix} \frac{\partial h_1^{-1}(u_1, u_2)}{\partial u_1} & \frac{\partial h_1^{-1}(u_1, u_2)}{\partial u_2} \\ \frac{\partial h_2^{-1}(u_1, u_2)}{\partial u_1} & \frac{\partial h_2^{-1}(u_1, u_2)}{\partial u_2} \end{vmatrix} = \begin{vmatrix} u_2 & u_1 \\ 0 & 1 \end{vmatrix} = u_2(1) - u_1(0) = u_2.$$

Therefore, the joint pdf of $(U_1, U_2)$, where nonzero, is

$$f_{U_1, U_2}(u_1, u_2) = f_{Y_1, Y_2}(h_1^{-1}(u_1, u_2), h_2^{-1}(u_1, u_2))|J| = f_{Y_1, Y_2}(u_1 u_2, u_2)|u_2| = 8(u_1 u_2) u_2 \times u_2 = 8u_1 u_2^3.$$
Summarizing, the joint pdf of \((U_1, U_2)\) is

\[
f_{U_1,U_2}(u_1, u_2) = \begin{cases} 
8u_1u_2^3, & 0 \leq u_1 \leq 1, 0 \leq u_2 \leq 1 \\
0, & \text{otherwise}
\end{cases}
\]

(b) Note that we can write

\[
f_{U_1,U_2}(u_1, u_2) = 8u_1u_2^3 = 2u_1 \times 4u_2^3 = f_{U_1}(u_1)f_{U_2}(u_2).
\]

We recognize

\[
f_{U_1}(u_1) = \begin{cases} 
2u_1, & 0 \leq u_1 \leq 1 \\
0, & \text{otherwise}
\end{cases}
\]

and

\[
f_{U_2}(u_2) = \begin{cases} 
4u_2^3, & 0 \leq u_2 \leq 1 \\
0, & \text{otherwise}
\end{cases}
\]

as beta pdfs. Specifically, \(U_1 \sim \text{beta}(2, 1)\) and \(U_2 \sim \text{beta}(4, 1)\). Because the joint pdf can be written as the product of the marginal pdfs, we know \(U_1 \perp \perp U_2\).