1. (a) You can use the transformation method or the cdf technique. Here are both solutions.

**Transformation method:** Note that \( u = h(r) = 10000e^r \) is a strictly increasing function over \([0.02, 0.04]\), the support of \( R \). Therefore, the support of \( U = h(R) = 10000e^R \) is

\[
R_U = \{ u : 10000e^{0.02} \leq u \leq 10000e^{0.04} \}.
\]

Let’s find the inverse transformation:

\[
u = h(r) = 10000e^r \iff \frac{u}{10000} = e^r \iff \ln\left(\frac{u}{10000}\right) = r = h^{-1}(u).
\]

The derivative of the inverse transformation is

\[
\frac{d}{du}h^{-1}(u) = \left|\frac{d}{du}\ln\left(\frac{u}{10000}\right)\right| = 50 \cdot \frac{1}{u}.
\]

Therefore, for \(10000e^{0.02} \leq u \leq 10000e^{0.04}\), the pdf of \(U\) is

\[
f_U(u) = \begin{cases} 
50 \cdot \frac{1}{u}, & 10000e^{0.02} \leq u \leq 10000e^{0.04} \\
0, & \text{otherwise.}
\end{cases}
\]

**CDF technique:** The cdf of \( R \sim \mathcal{U}(0.02, 0.04) \) is

\[
F_R(r) = \begin{cases} 
0, & r < 0.02 \\
\frac{r - 0.02}{0.02}, & 0.02 \leq r \leq 0.04 \\
1, & r > 0.04.
\end{cases}
\]

Therefore, for \(10000e^{0.02} \leq u \leq 10000e^{0.04}\), the cdf of \( U = h(R) = 10000e^R \) is

\[
F_U(u) = P(U \leq u) = P(10000e^R \leq u) = P\left(R \leq \ln\left(\frac{u}{10000}\right)\right) = F_R\left(\ln\left(\frac{u}{10000}\right)\right) = \frac{\ln\left(\frac{u}{10000}\right) - 0.02}{0.02}.
\]

Therefore, for \(10000e^{0.02} \leq u \leq 10000e^{0.04}\), the pdf of \(U\) is

\[
f_U(u) = \frac{d}{du}F_U(u) = \frac{d}{du}\left[\ln\left(\frac{u}{10000}\right) - 0.02\right] = 50 \cdot \frac{1/10000}{u/10000} = \frac{50}{u}.
\]

Summarizing,

\[
f_U(u) = \begin{cases} 
50 \cdot \frac{1}{u}, & 10000e^{0.02} \leq u \leq 10000e^{0.04} \\
0, & \text{otherwise.}
\end{cases}
\]

(b) You could calculate

\[
E(U) = E(10000e^R)
\]
directly by using the pdf of $R$ (by appealing to the Law of the Unconscious Statistician). Note that

$$E(10000e^R) = \int_{\mathbb{R}} 10000e^r f_R(r) \, dr = \int_{0.02}^{0.04} 10000e^r \times 50 \, dr$$

$$= 50000e^{r|_{0.04}^{0.02}} = 50000(e^{0.04} - e^{0.02}) = 10304.72.$$ 

You could also calculate $E(U) = E(10000e^R)$ directly by using the pdf of $U$. We have

$$E(U) = \int_{\mathbb{R}} uf_U(u) \, du = \int_{10000e^{0.02}}^{10000e^{0.04}} u \left( \frac{50}{u} \right) \, du$$

$$= 50u|_{10000e^{0.02}}^{10000e^{0.04}} = 50(10000)(e^{0.04} - e^{0.02}) = 10304.72.$$ 

2. (a) The mgf of $Y \sim \text{exponential}(1)$ is

$$m_Y(t) = \frac{1}{1 - t},$$

for $t < 1$. The mgf of $T = Y_1 + Y_2 + Y_3 + Y_4$ is

$$m_T(t) = [m_Y(t)]^4 = \left( \frac{1}{1 - t} \right)^4,$$

for $t < 1$. We recognize this as the mgf of a gamma random variable with shape $\alpha = 4$ and scale $\beta = 1$. Because mgfs are unique, $T \sim \text{gamma}(4, 1)$.

(b) The pdf of $Y_{(4)}$ is

$$f_{Y_{(4)}}(y) = 4f_Y(y)[F_Y(y)]^{4-1},$$

where

$$F_Y(y) = \left\{ \begin{array}{ll}
0, & y \leq 0 \\
1 - e^{-y}, & y > 0
\end{array} \right.$$ 

is the cdf of $Y \sim \text{exponential}(1)$. Therefore, for $y > 0$, we have

$$f_{Y_{(4)}}(y) = 4e^{-y}(1 - e^{-y})^3.$$ 

Summarizing,

$$f_{Y_{(4)}}(y) = \left\{ \begin{array}{ll}
4e^{-y}(1 - e^{-y})^3, & y > 0 \\
0, & \text{otherwise}
\end{array} \right.$$ 

3. You can use the cdf technique or a bivariate transformation. Here are both solutions.

**CDF technique:** The bivariate support of $Y_1$ and $Y_2$ is $R = \{(y_1, y_2) : 0 < y_1 < y_2 < 1\}$; see next page (left). The joint pdf $f_{Y_1, Y_2}(y_1, y_2)$ is a three-dimensional function which takes the value $6(1-y_2)$ over this region and is otherwise equal to zero. First note that

$$0 < y_1 < y_2 < 1 \implies u = \frac{y_1}{y_2} \in (0, 1).$$
Therefore, the support of

\[ U = \frac{Y_1}{Y_2} \]

is \( R_U = \{ u : 0 < u < 1 \} \). For \( 0 < u < 1 \), the cdf of \( U \) is

\[
F_U(u) = P(U \leq u) = P\left( \frac{Y_1}{Y_2} \leq u \right) = \int_{(y_1, y_2) \in B} f_{Y_1, Y_2}(y_1, y_2) dy_1 dy_2 = \int_{(y_1, y_2) \in B} 6(1-y_2)dy_1 dy_2,
\]

where the set \( B = \{(y_1, y_2) : 0 < y_1 < y_2 < 1, \frac{y_1}{y_2} \leq u \} \) is shown above (right). Note that the boundary of \( B \) is

\[
\frac{y_1}{y_2} = u \implies y_2 = \frac{y_1}{u},
\]

a linear function of \( y_1 \) with slope \( 1/u \) and intercept 0. This boundary line is shown above and the set \( B \) is shown shaded (right). The limits to calculate the double integral above come from this picture. For \( 0 < u < 1 \), we have

\[
F_U(u) = \int_{y_2=0}^{1} \int_{y_1=0}^{u y_2} 6(1-y_2)dy_1 dy_2 = \int_{y_1=0}^{1} 6(1-y_2) \left( \int_{y_1=0}^{u y_2} dy_1 \right) dy_2
\]

\[= \int_{y_2=0}^{1} 6(1-y_2) y_2 dy_2 = u \int_{y_2=0}^{1} 6y_2(1-y_2) dy_2 = u. \]
The last integral equals 1 because $6y_2(1 - y_2)$ is the $\text{beta}(2, 2)$ pdf and we are integrating it over $(0, 1)$. We have shown

$$F_U(u) = \begin{cases} 0, & u \leq 0 \\ u, & 0 < u < 1 \\ 1, & u \geq 1. \end{cases}$$

This is the cdf of $Y \sim \mathcal{U}(0, 1)$. Thus, we are done.

**Bivariate transformation:** To perform a bivariate transformation, let

$$U_1 = h_1(Y_1, Y_2) = \frac{Y_1}{Y_2}$$

$$U_2 = h_2(Y_1, Y_2) = Y_1 \quad \leftarrow \text{dummy variable}$$

We will perform a bivariate transformation to obtain the joint pdf $f_{U_1, U_2}(u_1, u_2)$. We will then integrate $f_{U_1, U_2}(u_1, u_2)$ over $u_2$ to obtain the marginal pdf $f_{U_1}(u_1)$.

It is easy to see this transformation is 1:1. Suppose $h(y_1, y_2) = h(y_1^*, y_2^*)$, where

$$h \left( \begin{array}{c} y_1 \\ y_2 \end{array} \right) = \left( \begin{array}{c} h_1(y_1, y_2) \\ h_2(y_1, y_2) \end{array} \right) = \left( \begin{array}{c} \frac{y_1}{y_2} \\ y_1 \end{array} \right).$$

It immediately follows $y_1 = y_1^*$ (from the second equation) and therefore $y_2 = y_2^*$ from the first.

What is the support of $U_1$ and $U_2$? Clearly,

$$0 < y_1 < 1 \implies 0 < u_2 < 1.$$ 

In addition,

$$0 < y_1 < y_2 < 1, \quad \implies \quad 0 < u_1 = \frac{y_1}{y_2} < 1.$$ 

However, also note that

$$u_2 = y_1 < \frac{y_1}{y_2} = u_1$$

because $y_2 < 1$. Therefore, the support of $U_1$ and $U_2$ is

$$R_{U_1, U_2} = \{ (u_1, u_2) : 0 < u_2 < u_1 < 1 \}.$$ 

This region is shown at the top of the next page. Next, we find the inverse transformation. Clearly,

$$y_1 = h_1^{-1}(u_1, u_2) = u_2.$$ 

Therefore,

$$u_1 = \frac{y_1}{y_2} \implies y_2 = \frac{y_1}{u_1} \implies y_2 = \frac{u_2}{u_1} = h_2^{-1}(u_1, u_2).$$ 

The inverse transformation is given by

$$y_1 = h_1^{-1}(u_1, u_2) = u_2$$

$$y_2 = h_2^{-1}(u_1, u_2) = \frac{u_2}{u_1}.$$
The Jacobian is
\[
J = \det \begin{vmatrix}
\frac{\partial h^{-1}_1(u_1, u_2)}{\partial u_1} & \frac{\partial h^{-1}_1(u_1, u_2)}{\partial u_2} \\
\frac{\partial h^{-1}_2(u_1, u_2)}{\partial u_1} & \frac{\partial h^{-1}_2(u_1, u_2)}{\partial u_2}
\end{vmatrix}
= \det \begin{vmatrix}
0 & 1 \\
-\frac{u_2}{u_1^2} & \frac{1}{u_1}
\end{vmatrix}
= 0 \left( \frac{1}{u_1} \right) - 1 \left( -\frac{u_2}{u_1^2} \right) = \frac{u_2}{u_1^2}.
\]

Therefore, the joint pdf of \((U_1, U_2)\), where nonzero, is
\[
f_{U_1, U_2}(u_1, u_2) = f_{Y_1, Y_2}(h^{-1}_1(u_1, u_2), h^{-1}_2(u_1, u_2)) | J |
= f_{Y_1, Y_2} \left( \frac{u_2}{u_1} \right) \left| \frac{u_2^2}{u_1^3} \right|
= 6 \left( 1 - \frac{u_2}{u_1} \right) \times \frac{u_2}{u_1^2} = 6 \left( \frac{u_2}{u_1^2} - \frac{u_2^3}{u_1^3} \right).
\]

Summarizing,
\[
f_{U_1, U_2}(u_1, u_2) = \begin{cases} 
6 \left( \frac{u_2}{u_1^2} - \frac{u_2^3}{u_1^3} \right), & 0 < u_2 < u_1 < 1 \\
0, & \text{otherwise.}
\end{cases}
\]

This completes the bivariate transformation. The marginal pdf of \(U_1\), for \(0 < u_1 < 1\), is
\[
f_{U_1}(u_1) = \int_{\mathbb{R}} f_{U_1, U_2}(u_1, u_2) du_2
= \int_{u_2=0}^{u_1} 6 \left( \frac{u_2}{u_1^2} - \frac{u_2^3}{u_1^3} \right) du_2
= 6 \left( \frac{u_2^2}{2u_1^2} - \frac{u_2^3}{3u_1^3} \right) \bigg|_{u_2=0}^{u_1}
= 6 \left( \frac{u_1^2}{2u_1^2} - \frac{u_1^3}{3u_1^3} \right) = 6 \left( \frac{1}{2} - \frac{1}{3} \right) = 1.
\]

Summarizing,
\[
f_{U_1}(u_1) = \begin{cases} 
1, & 0 < u_1 < 1 \\
0, & \text{otherwise.}
\end{cases}
\]

This is the \(U(0, 1)\) pdf. Thus, we are done.
4. (a) \( Q_1 \sim \mathcal{N}(0, 1); \ Q_2 \sim \chi^2(n-1). \)

(b) We know 
\[
Q_1 \sim \mathcal{N}(0, 1) \implies Q_1^2 \sim \chi^2(1).
\]
We also know \( Q_1 \perp Q_2. \) This is true because \( Q_1 \) is a function of \( Y \) and \( Q_2 \) is a function of \( S^2 \) (and \( Y \perp S^2 \)). Therefore, the mgf of the sum \( Q_1^2 + Q_2 \) is the product of the respective mgfs; i.e.,
\[
m_{Q_1^2}(t)m_{Q_2}(t) = \left( \frac{1}{1 - 2t} \right)^{\frac{1}{2}} \left( \frac{1}{1 - 2t} \right)^{\frac{n-1}{2}} = \left( \frac{1}{1 - 2t} \right)^{\frac{n}{2}}.
\]
We recognize this as the \( \chi^2(n) \) mgf. Because mgfs are unique, we have \( Q_1^2 + Q_2 \sim \chi^2(n). \)

(c) We’ll start by following the hint. We know \( Y_{n+1} \sim \mathcal{N}(\mu, \sigma^2) \) and 
\[
(\mu, \sigma^2) = (\mu, \sigma^2(n)).
\]
We know \( Y_{n+1} - \overline{Y} \) is a linear combination of normal random variables, so it is normally distributed too. Let’s calculate the mean and variance of \( Y_{n+1} - \overline{Y} \); we have 
\[
E(Y_{n+1} - \overline{Y}) = E(Y_{n+1}) - E(\overline{Y}) = \mu - \mu = 0
\]
and 
\[
V(Y_{n+1} - \overline{Y}) = V(Y_{n+1}) + V(\overline{Y}) - 2 \text{Cov}(Y_{n+1}, \overline{Y}) = \sigma^2 + \frac{\sigma^2}{n} = \sigma^2 \left( 1 + \frac{1}{n} \right).
\]
The covariance term above is zero because \( Y_{n+1} \) is independent of \( Y_1, Y_2, \ldots, Y_n \) (and hence \( Y_{n+1} \) is independent of any function of \( Y_2, \ldots, Y_n \), like \( \overline{Y} \)). Therefore,
\[
Y_{n+1} - \overline{Y} \sim \mathcal{N}
\left(0, \sigma^2 \left( 1 + \frac{1}{n} \right) \right) \implies Z = \frac{Y_{n+1} - \overline{Y}}{\sqrt{\sigma^2 \left( 1 + \frac{1}{n} \right)}} \sim \mathcal{N}(0, 1).
\]
Now, consider 
\[
Q_2 = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1).
\]
Note that \( Z \perp Q_2. \) This is true because both \( Y_{n+1} \) and \( \overline{Y} \) are independent of \( S^2 \). Therefore, we can create a \( t \) random variable by taking 
\[
\frac{Z}{\sqrt{Q_2/(n-1)}} = \frac{Y_{n+1} - \overline{Y}}{S \sqrt{1 + \frac{1}{n}}} \sim t(n-1).
\]

5. (a) We can find the pdf of \( U = h(Y) = Y + 2 \) by using a transformation. Note that \( h(y) = y + 2 \) is a linear function; hence it is 1:1. Also, note that 
\[
y > 0 \implies u = y + 2 > 2.
\]
Therefore, the support of $U$ is $R_U = \{u : u > 2\}$. The inverse transformation is found as follows:
\[ u = h(y) = y + 2 \implies y = h^{-1}(u) = u - 2. \]

The derivative of the inverse transformation is
\[ \frac{d}{du} h^{-1}(u) = \frac{d}{du}(u - 2) = 1. \]

Therefore, the pdf of $U$, for $u > 2$, is given by
\[
\begin{align*}
  f_U(u) &= f_Y(h^{-1}(u)) \left| \frac{d}{du} h^{-1}(u) \right| \\
  &= \frac{24}{(u - 2 + 2)^4} \times 1 = \frac{24}{u^4}.
\end{align*}
\]

Summarizing, the pdf of $U$ is
\[
  f_U(u) = \begin{cases} 
  \frac{24}{u^4}, & u > 2 \\
  0, & \text{otherwise}.
\end{cases}
\]

(b) To approximate $P(\overline{U} > 5)$, we will use the CLT. The (approximate) sampling distribution of $\overline{U}$ conferred by the CLT is
\[
\overline{U} \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right),
\]
where $\mu = E(U)$ and $\sigma^2 = V(U)$. Let’s calculate these. The mean of $U$ is
\[
E(U) = \int_R u f_U(u) du = \int_2^\infty u \left(\frac{24}{u^4}\right) du = \left. \frac{24}{u^3} \right|_2^\infty = 12 \left(\frac{1}{4} - 0\right) = 3.
\]

The second moment of $U$ is
\[
E(U^2) = \int_R u^2 f_U(u) du = \int_2^\infty u^2 \left(\frac{24}{u^4}\right) du = \left. \frac{24}{u^2} \right|_2^\infty = 24 \left(\frac{1}{2} - 0\right) = 12.
\]

Therefore, the variance of $U$ is
\[
V(U) = E(U^2) - [E(U)]^2 = 12 - (3)^2 = 3.
\]

The approximate sampling distribution of $\overline{U}$ (based on an iid sample of size $n = 20$) is
\[
\overline{U} \sim \mathcal{N}\left(3, \frac{3}{20}\right).
\]

Therefore,
\[
P(\overline{U} > 5) = P\left(\frac{\overline{U} - 3}{\frac{3}{\sqrt{20}}} > \frac{5 - 3}{\sqrt{\frac{3}{20}}}\right) \approx P(Z > 5.16),
\]

where $Z \sim \mathcal{N}(0, 1)$. Therefore, $P(\overline{U} > 5)$ can be approximated by finding the right-tail probability $P(Z > 5.16)$ on the $\mathcal{N}(0, 1)$ pdf. This probability is extremely small ($\approx 0.000000123$), so it is highly unlikely the average hospitalization period will exceed 5 days.