GROUND RULES:

- This exam contains 5 questions. Each question is worth 10 points.
- I am only requiring you to hand in the first 4 problems. Therefore, this exam is worth 40 points.
- Question 5 is worth 10 points **extra credit**. Therefore, it is possible for you to score higher than 40 points on this exam (the highest one can score is 50 points).
- Print your name at the top of this page in the upper right hand corner.
- This is a closed-book and closed-notes exam. You may use a calculator if you wish. Show all of your work and explain all of your reasoning!
- Any discussion or otherwise inappropriate communication between examinees, as well as the appearance of any unnecessary material, will be dealt with severely.
- You have 55 minutes to complete this exam. GOOD LUCK!

HONOR PLEDGE FOR THIS EXAM:

After you have finished the exam, please read the following statement and sign your name below it.

> *I promise that I did not discuss any aspect of this exam with anyone other than the instructor, that I neither gave nor received any unauthorized assistance on this exam, and that the work presented herein is entirely my own.*

1. Suppose that $Y_1, Y_2, ..., Y_n$ is an iid $\mathcal{N}(0, \sigma^2)$ sample, where $\sigma^2 > 0$ is unknown. We are interested in testing

$$H_0 : \sigma^2 = 1$$
$$\text{versus}$$
$$H_a : \sigma^2 > 1.$$

To perform the test, suppose we use the test statistic

$$T = \sum_{i=1}^{n} Y_i^2$$

and the rejection region $\text{RR} = \{t : t > k\}$, where k is a constant.
(a) Determine the value of k that makes RR have Type I Error probability equal to α.
(b) Using your answer from part (b), derive an expression for

$$\beta(\sigma_a^2) = P(\text{Type II Error}|\sigma^2 = \sigma_a^2),$$

where $\sigma_a^2 > 1$. Show that $\beta(\sigma_a^2)$ is a decreasing function of σ_a^2.

2. Suppose that $Y_1, Y_2, ..., Y_{12}$ is an iid sample of Poisson observations with mean θ. Consider testing

$$H_0 : \theta = 1/2$$
versus
$$H_a : \theta < 1/2.$$

Suppose that we decide to reject H_0 when

$$T = Y_1 + Y_2 + \cdots + Y_{12} \leq 2.$$

(a) Let $K(\theta)$ denote the power function for this test. Compute $K(1/2)$, $K(1/3)$, $K(1/4)$, $K(1/6)$, and $K(1/12)$. Then, sketch a graph of $K(\theta)$.
(b) What is $\alpha = P(\text{Type I Error})$?

3. A high quality bullet proof glass will stop a bullet 80 percent of the time. A low quality glass will stop a bullet only 60 percent of the time.

You have received a shipment of bullet proof glass, but you do not know whether it is low or high quality. To test which glass type you have, you have decided to experiment by shooting bullets at n sheets of glass. Your decision as to whether the glass is low or high quality should be based on the number (or proportion) of the n sheets of glass that stop the bullet.

Specifically, your job is to determine the smallest number n of sheets of glass you will have to shoot so that

- if the glass is of high quality, the probability of wrongly concluding that it is low quality is $\alpha \approx 0.05$.
- if the glass is of low quality, the probability you incorrectly declare it to be of high quality is $\beta \approx 0.2$.

What is the smallest sample size n that meets these requirements? If you can not find a numerical answer for n, you can outline an approach that would lead one to the correct answer; e.g., from solving an equation (or equations) for n perhaps.

4. Suppose that $Y_1, Y_2, ..., Y_5$ is an iid sample of size $n = 5$ from

$$f_Y(y;\theta) = \begin{cases} \theta e^{-\theta y}, & y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Find the uniformly most powerful (UMP) level $\alpha = 0.10$ rejection region to test

$$H_0 : \theta = 1$$
versus
$$H_a : \theta < 1.$$

Your rejection region should not include any unknown constants. If it includes a quantile from a well known distribution, use properly defined notation to identify it.

5. **Extra Credit.** Suppose that we have two independent samples:

$$\begin{aligned} \text{Sample 1}: &\quad Y_{11}, Y_{12}, ..., Y_{1n_1} \sim \text{iid } \mathcal{N}(\mu_1, \sigma^2) \\ \text{Sample 2}: &\quad Y_{21}, Y_{22}, ..., Y_{2n_2} \sim \text{iid } \mathcal{N}(\mu_2, c\sigma^2), \end{aligned}$$

where the constant c is known.

(a) Derive a level α hypothesis test for

$$H_0 : \sigma^2 = 1$$
versus
$$H_a : \sigma^2 \neq 1$$

using all the observations from both samples. I want you to do this **two ways**, by identifying a suitable test statistic that follows a

(i) χ^2 distribution

(ii) F distribution.

For each case, your derivation should specify a test statistic and the rejection region.
(b) If you had to pick between the two level α tests you derived in part (a), which would you choose? Why?