1. Recall that if X has a beta(α, β) distribution, then the probability density function (pdf) of X is

$$f_X(x|\alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} I(0 < x < 1).$$

(a) Show that $\{f_X(x|\alpha, \beta); \alpha > 0, \beta > 0\}$ is a two-parameter exponential family. Is this a full or curved family?
(b) For this part only, suppose that $\alpha > 1$ and $\beta > 1$. Find the mode of X, that is, find the value of x that maximizes $f_X(x|\alpha, \beta)$.
(c) Derive the pdf of $Y = 1 - X$.

2. A philanthropist decides to choose s persons at random and to invite them to a series of parties over the year. A party is given on a particular day if at least one person (among the s) has a birthday on that day (multiple parties cannot occur on the same day). Assume, for simplicity, that there are exactly 365 days in the year and that birthdays occur at random across days. Define

$$X_i = \begin{cases} 1, & \text{if a party is given on day } i, \\ 0, & \text{otherwise,} \end{cases}$$

for $i = 1, 2, ..., 365$. Therefore, $X_1, X_2, ..., X_{365}$ is a sequence of 0-1 random variables.
(a) What is the distribution of X_1?
(b) Compute $\text{cov}(X_i, X_j)$, where $i \neq j$.
(c) Let T denote the number of parties that will be given over the year. Find the mean and variance of T.

3. Suppose that U_1 and U_2 are independent random variables with $U_1 \sim \chi^2_{n_1}(\lambda)$ and $U_2 \sim \chi^2_{n_2}$. That is, U_1 is distributed as noncentral χ^2 with degrees of freedom $n_1 > 0$ and noncentrality parameter $\lambda > 0$, and U_2 is distributed as (central) χ^2 with degrees of freedom $n_2 > 0$.
(a) Recall that the distribution of U_1 arises as a mixture in the following hierarchy:

$$\begin{aligned} U_1|Y &\sim \chi^2_{n_1+2Y} \\ Y &\sim \text{Poisson}(\lambda). \end{aligned}$$

Show that $E(U_1) = n_1 + 2\lambda$ and $\text{var}(U_1) = 2n_1 + 8\lambda$.
(b) We know the random variable

$$W = \frac{U_1/n_1}{U_2/n_2}$$

has a non-central F distribution, as described in Homework 11. Show that

$$E(W) = \frac{n_2}{n_2 - 2}\left(1 + \frac{2\lambda}{n_1}\right)$$

for $n_2 > 2$. Make sure you explain why $n_2 > 2$.

4. Let $F_X : \mathbb{R} \to [0, 1]$ and $F_Y : \mathbb{R} \to [0, 1]$ be univariate cumulative distribution functions (cdfs) and suppose $-1 \leq \alpha \leq 1$. Define $F^{(\alpha)}_{X,Y} : \mathbb{R}^2 \to [0, 1]$ by

$$F^{(\alpha)}_{X,Y}(x, y) = F_X(x)F_Y(y)\{1 + \alpha[1 - F_X(x)][1 - F_Y(y)]\}.$$

The collection $\{F_{X,Y}^{(\alpha)} : -1 \leq \alpha \leq 1\}$ is called the *Farlie-Morgenstern family* of bivariate cdfs corresponding to F_X and F_Y.
(a) Starting with the expression for $F_{X,Y}^{(\alpha)}(x,y)$, show that the marginal cdfs of X and Y are given by $F_X(x)$ and $F_Y(y)$, respectively.
(b) What value of α corresponds to X and Y being independent? Explain.
(c) As a special case, suppose that X and Y are each distributed as exponential with mean 1. In this case, for any $\alpha \in [-1,1]$, show that the joint probability density function of $(X,Y)'$ is

$$f_{X,Y}^{(\alpha)}(x,y) = \{1 + \alpha[(1-2e^{-x})(1-2e^{-y})]\}e^{-(x+y)}I(x>0)I(y>0).$$

5. Suppose that $X_1, X_2, ..., X_n$ is an iid sample from

$$f_X(x|\mu) = e^{-(x-\mu)}I(x>\mu).$$

(a) Show that $\{f_X(x|\mu); -\infty < \mu < \infty\}$ is a location family.
(b) Show that the probability density function of the minimum order statistic $X_{(1)}$ is

$$f_{X_{(1)}}(x|\mu) = ne^{-n(x-\mu)}I(x>\mu).$$

(c) Show that $X_{(1)}$ converges in probability to μ.