1. Suppose that $S = \{\omega \in \mathbb{R} : -1 < \omega < 2\}$ and let $\mathcal{B} = \{B \cap S : B \in \mathcal{B}(\mathbb{R})\}$ denote the collection of Borel sets on S. Define the sequence of events

$$A_n = \begin{cases} (-1/n, 0], & \text{if } n \text{ is odd;} \\ (0, 1 + 1/n], & \text{if } n \text{ is even.} \end{cases}$$

(a) True or False: $\{A_n, n = 1, 2, ..., \}$ partitions S.
(b) Does $\lim_{n\to\infty} A_n$ exist? If so, show that it does. If not, show why not.
(c) Suppose that P is a set function on $(S, \mathcal{B})$ defined by

$$P(A) = \int_A \frac{2}{9}(x+1)dx$$

for any $A \in \mathcal{B}$. Show that P satisfies the Kolmolgorov Axioms.

2. Suppose $(S, \mathcal{B}, P)$ is a probability space and let A and B be measurable events; i.e., $A \in \mathcal{B}$ and $B \in \mathcal{B}$. Treat parts (a), (b), and (c) separately; they are not related in any way.
(a) If $P(A \cap B) = 0$ and $P(A) = 1$, show that $P(B) = 0$.
(b) If $P(A) > 0$, $P(B) > 0$, and $P(A) < P(A|B)$, show that $P(B) < P(B|A)$.
(c) If A and B are independent, show that A^c and B^c are independent.

3. There were five accidents in a town during a seven-day period (one week). Would you be surprised if all five accidents occurred on the same day? What if each of the five accidents occurred on a different day? Answer these questions by describing a relevant probability space $(S, \mathcal{B}, P)$ and making relevant calculations.

4. Suppose that X is a discrete random variable with probability mass function

$$f_X(x) = \begin{cases} (\frac{1}{2})^{x+1}, & x = 0, 1, 2, 3, ..., \\ 0, & \text{otherwise.} \end{cases}$$

(a) Derive $E(X)$ and $\text{var}(X)$. Don't just state the answers (if you know them).
(b) Derive the probability mass function of $Y = X/(X + 1)$.

5. A continuous random variable X is said to have a *Cauchy distribution* if its probability density function is

$$f_X(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty.$$

Define $Y = -X$ and $Z = 1/X$. Show that X, Y, and Z all have the same distribution.

6. A random variable X is said to be *bounded* if there exists $M > 0$ such that

$$P_X(|X| \le M) = P_X(-M \le X \le M) = 1.$$

(a) Suppose X is a bounded continuous random variable with probability density function (pdf) $f_X(x)$. Prove that $E(|X|) \le M$.

(b) Suppose X is a bounded continuous random variable with pdf $f_X(x)$. Prove that the moment generating function of X exists for all $t \in \mathbb{R}$.

Hint: For both parts, you can assume that $f_X(x) = 0$ if $x \notin [-M, M]$. Explain why.