1. Suppose that $X$ is a continuous random variable with probability density function (pdf)

$$f_X(x|\theta) = \theta x^{-(\theta+1)} I(x > 1),$$

where $\theta > 0$.
(a) Calculate an exact expression for the tail probability $P(X > c)$, for $c > 1$.
(b) Calculate the upper bound on $P(X > c)$, $c > 1$, provided by Markov's Inequality. Note any restrictions on $\theta$ that are needed for the upper bound to be applicable.

2. In each part below, give an example of a parametric family $\{f_X(x|\boldsymbol{\theta}); \boldsymbol{\theta} \in \Theta\}$ that satisfies the stated conditions. Specify the family, the parameter space $\Theta$, and the support $\mathcal{X}$. Prove any claims you make.
(a) a family that is a single parameter exponential family (i.e., $d = 1$) and also a scale family.
(b) a family that is a two-parameter full exponential family (i.e., $d = k = 2$) and also a location-scale family.
(c) a two-parameter full exponential family with $\boldsymbol{\theta} = (\theta_1, \theta_2)$ that contains a subfamily, with $\theta_2 = g(\theta_1)$, that is a full one-parameter exponential family.

3. Consider the random vector $(X, Y)'$ with probability density function (pdf)

$$f_{X,Y}(x, y) = 2x^{-1}e^{-2x} I(0 < y < x < \infty).$$

(a) Verify that this is a valid pdf.
(b) Calculate $P(Y > X/2, X < 1)$.
(c) Calculate the correlation of $X$ and $Y$.

4. Suppose that $(X, Y)'$ has the following probability density function (pdf)

$$f_{X,Y}(x, y) = (xy)^{-2} I(x > 1) I(y > 1).$$

(a) Are $X$ and $Y$ independent? Explain.
(b) Find the joint pdf of $U = XY$ and $V = X/Y$.
(c) Find the marginal pdfs of $U$ and $V$. Are $U$ and $V$ independent?

5. Suppose that $X_1, X_2, ..., X_n$ are mutually independent random variables. Each random variable, marginally, follows a standard normal distribution.
(a) Derive the distribution of $S = \frac{1}{n}(X_1 + X_2 + \cdots + X_n)$.
(b) Derive the distribution of $T = X_1^2 + X_2^2 + \cdots + X_n^2$.
(c) Derive the moment-generating functions (mgf) of

$$U = \sqrt{n}S \qquad \text{and} \qquad V = \frac{T - n}{\sqrt{2n}}.$$

(Derive each one separately; I am not asking for a "joint" mgf here). What does each mgf converge to as $n \to \infty$?

6. Suppose that $X$ and $Y$ are random variables with finite means and variances. Suppose that we want to predict $Y$ as a linear function of $X$. That is, we are interested only in functions of the form $Y = \beta_0 + \beta_1 X$, for fixed constants $\beta_0$ and $\beta_1$. Define the *mean squared error of prediction* by

$$Q(\beta_0, \beta_1) \equiv E\{[Y - (\beta_0 + \beta_1 X)]^2\}.$$

(a) Show that $Q(\beta_0, \beta_1)$ is minimized when

$$\beta_1 = \rho \left(\frac{\sigma_Y}{\sigma_X}\right)$$

and

$$\beta_0 = E(Y) - \beta_1 E(X),$$

where $\rho = \text{corr}(X, Y)$.
(b) Calculate the values of $\beta_0$ and $\beta_1$ for the bivariate distribution in Problem 3. Does $\beta_0 + \beta_1 X$ equal $E(Y|X)$ in that problem? Comment.